Operatori, svojstveni problem, reprezentacije

1. Hilbertov prostor ($\mathcal{H}$)

beskonačnodimenzioni, unitarni, separabilni i
→ postoji ONB

kompletni vektorski prostor sa skalarnim proizvodom
konvergencija reda

$$\psi \in \mathcal{H} \Rightarrow |\psi\rangle = \sum_{i=1}^{\infty} c_i |\varphi_i\rangle$$

Dirakova simbolika

diskretni bazis u $\mathcal{H}$

$$\langle \varphi_i | \varphi_j \rangle = \delta_{ij}$$

2. Operatori

a) Ermitski $\hat{A}^+ = \hat{A}$ $\qquad \hat{A}^+ = (\hat{A}^T)^*$

↳ KM opservable

b) Unitarni $\hat{U}^+ = \hat{U}^{-1}$ $\qquad \hat{U}\hat{U}^+ = \hat{U}^+\hat{U} = \hat{I}$

identični operator

$[\hat{A}, \hat{B}]$ – komutator $\to \hat{A}\hat{B} - \hat{B}\hat{A}$

$\{\hat{A}, \hat{B}\}$ – antikomutator $\to \hat{A}\hat{B} + \hat{B}\hat{A}$

$[\hat{A}, \hat{B}] \neq 0 \Rightarrow$ Hajzenbergove relacije neodređenosti

3. Uopšteni Hilbertov prostor $U(\mathcal{H})$

$$\left.\begin{array}{l} |a\rangle \in U(\mathcal{H}) \\ |a'\rangle \in U(\mathcal{H}) \end{array}\right\} \quad \langle a | a' \rangle = \delta(a - a') = \begin{cases} \infty, & a = a' \\ 0, & a \neq a' \end{cases}$$

Primer: $\hat{p}_x |p_x\rangle = p_x |p_x\rangle \qquad p_x \in (-\infty, +\infty)$

$$\langle p_x | p_x' \rangle = \delta(p_x - p_x')$$

1. Svojstveni problem opservabli

$$\hat{A}|\varphi_i\rangle = a_i|\varphi_i\rangle$$

$\{a_i\}$ – diskretan (prebrojiv) spektar svojstvenih vrednosti

Spektralna forma $\hat{A} = \sum_i a_i |\varphi_i\rangle\langle\varphi_i| = \sum_i a_i \hat{P}_i$

$$\hat{A}|a\rangle = a|a\rangle$$

$a \in (\alpha, \beta)$ kontinualni (neprebrojivi) spek. svoj. vred.

Spektralna forma operatora $\hat{A}$ sa mešovitom spektrom

$$\hat{A} = \sum_i a_i \hat{P}_i + \int_\alpha^\beta a |a\rangle\langle a| \, da$$

$\hat{P}_i$ – projektori $\quad \hat{P}_i\hat{P}_j = \hat{P}_i\delta_{ij}, \quad \hat{P}_i^2 = \hat{P}_i$

$\sum_i \hat{P}_i = \hat{I}$ razlaganje jedinice

$$\int_\alpha^\beta |a\rangle\langle a| \, da = \hat{I}$$

5. TENZORSKI proizvod Hilbertovih prostora

$$\mathcal{H}^{(\omega)} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$$

$$\mathcal{H}^{(r)} = \mathcal{H}^{(x)} \otimes \mathcal{H}^{(y)} \otimes \mathcal{H}^{(z)}$$

$$\hat{\vec{p}}|\vec{p}\rangle = \vec{p}|\vec{p}\rangle$$

$$\hat{\vec{p}}|\vec{p}\rangle = (\hat{p}_x\vec{e}_x + \hat{p}_y\vec{e}_y + \hat{p}_z\vec{e}_z)|\vec{p}\rangle$$

$$\hat{p}_x \otimes \hat{I}_y \otimes \hat{I}_z |p_x\rangle|p_y\rangle|p_z\rangle = \hat{p}_x|\vec{p}\rangle$$

$$\hat{\vec{p}}^2 = \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2 = \hat{p}_x^2 \otimes \hat{I}_y \otimes \hat{I}_z + \hat{I}_x \otimes \hat{p}_y^2 \otimes \hat{I}_z + \hat{I}_x \otimes \hat{I}_y \otimes \hat{p}_z^2$$

$$(\hat{A}_1 \otimes \hat{B}_2)(|\psi_1\rangle \otimes |\psi_2\rangle) = \hat{A}_1|\psi_1\rangle \otimes \hat{B}_2|\psi_2\rangle$$

Matematički podsetnik (Domaći)

Vektorski proizvod vektora u Euklidovom prostoru

$$\vec{A} \times \vec{B} = \varepsilon_{ijk} A_i B_j \vec{e}_k$$

i veza sa Levi-Civita simbolom

- Šta je to bazis?

To je skup LNZ vektora sa osobinom da je jedino $\vec{0}$ ortogonalan na sve vektore bazisa.

(Mekloren i Tejlor)

- Kako glasi razvoj u red f-je $f(x)$?

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

- Kako se rešava j-na

$$y''(x) + k^2 y(x) = 0$$

Predispitne obaveze
T. Mehanika
M. fizika 1

(MF)
Kolokvijum - 10 poena
Dolasci na vežbe 5 poena
Pismeni deo ispita 35 poena (3 zadatka)

1. Proračunati ili dokazati sledeće komutacione relacije (koje važe u bilo kojem Hilbertovom prostoru):

a) $[\hat{A},\hat{B}] = -[\hat{B},\hat{A}]$

b) $[\hat{A}+\hat{B},\hat{C}] = [\hat{A},\hat{C}] + [\hat{B},\hat{C}]$

c) $[a\hat{A}, b\hat{B}] = ab\,[\hat{A},\hat{B}]$

d) $[\hat{A},\hat{B}\hat{C}] = [\hat{A},\hat{B}]\hat{C} + \hat{B}[\hat{A},\hat{C}]$ BAC

e) $[\hat{A}\hat{B},\hat{C}] = [\hat{A},\hat{C}]\hat{B} + \hat{A}[\hat{B},\hat{C}]$ ACB

f) $[\hat{A},\hat{B}^n] = \sum\limits_{s=0}^{n-1} \hat{B}^s [\hat{A},\hat{B}]\, \hat{B}^{n-s-1}$

g) $[\hat{A},[\hat{B},\hat{C}]] + [\hat{B},[\hat{C},\hat{A}]] + [\hat{C},[\hat{A},\hat{B}]] = 0$

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a), b) $\Big\}$ sami, primenom definicije

$$[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

c) $[a\hat{A}, b\hat{B}] = ab\,\hat{A}\hat{B} - ab\,\hat{B}\hat{A} = ab\,[\hat{A},\hat{B}]$

d) L.S. $[\hat{A},\hat{B}\hat{C}] \overset{d}{=} \hat{A}\hat{B}\hat{C} - \hat{B}\hat{C}\hat{A} = \hat{A}\hat{B}\hat{C} - \hat{B}\hat{C}\hat{A} + \hat{B}\hat{A}\hat{C} - \hat{B}\hat{A}\hat{C} = (\hat{A}\hat{B}\hat{C} - \hat{B}\hat{A}\hat{C}) + (\hat{B}\hat{A}\hat{C} - \hat{B}\hat{C}\hat{A})$

$= (\hat{A}\hat{B} - \hat{B}\hat{A})\hat{C} + \hat{B}(\hat{A}\hat{C} - \hat{C}\hat{A})$

$= [\hat{A},\hat{B}]\hat{C} + \hat{B}[\hat{A},\hat{C}]$

e) $[\hat{A}\hat{B},\hat{C}] = \hat{A}\hat{B}\hat{C} - \hat{C}\hat{A}\hat{B} + \hat{A}\hat{C}\hat{B} - \hat{A}\hat{C}\hat{B}$

$$= \hat{A}(\hat{B}\hat{C}-\hat{C}\hat{B}) + (\hat{A}\hat{C}-\hat{C}\hat{A})\hat{B}$$

$$= \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B}$$

f) Dokaz matematičkom indukcijom

za $n=1 \Rightarrow s=0$ i tvrdenje je tačno

Neka je tačno za $n=k$, onda proverau da li je tačno za $n=k+1$

$$[\hat{A},\hat{B}^{k+1}] = [\hat{A},\hat{B}^k\hat{B}] \overset{(d)}{=} \underbrace{[\hat{A},\hat{B}^k]\hat{B} + \hat{B}^k[\hat{A},\hat{B}]}_{(D.S)_1}$$

$$[\hat{A},\hat{B}^{k+1}] = \sum_{s=0}^{k} \hat{B}^s[\hat{A},\hat{B}]\hat{B}^{k-s}$$

$$= \sum_{s=0}^{k-1} \hat{B}^s[\hat{A},\hat{B}]\hat{B}^{k-s} + \hat{B}^k[\hat{A},\hat{B}]\hat{B}^0$$

$$= \underbrace{\sum_{s=0}^{k-1} \hat{B}^s[\hat{A},\hat{B}]\hat{B}^{k-s-1}}\hat{B} + \hat{B}^k[\hat{A},\hat{B}]$$

$$= [\hat{A},\hat{B}^k]\hat{B} + \hat{B}^k[\hat{A},\hat{B}] = (D.S.)_2$$

$(D.S)_1 = (D.S)_2 \Rightarrow$ važi i za $n=k+1$

) za domaći, koristeći def. komutatora i pokazano za d)

2. Definicija izvoda operatora $\hat{A}$ po parametru λ je formalno identična definiciji izvoda u standardnoj analizi. Pravila izvoda moraju voditi računa o (ne)komutiranju operatora u sumi i uzastopnom delovanju operatora:

$$\frac{d(\hat{A}+\hat{B})}{d\lambda} = \frac{d\hat{A}}{d\lambda} + \frac{d\hat{B}}{d\lambda} \quad i \quad \frac{d(\hat{A}\hat{B})}{d\lambda} = \frac{d\hat{A}}{d\lambda}\hat{B} + \hat{A}\frac{d\hat{B}}{d\lambda}.$$

Na osnovi ovih pravila dokazati:

$$\frac{d\hat{A}^{-1}}{d\lambda} = -\hat{A}^{-1}\frac{d\hat{A}}{d\lambda}\hat{A}^{-1}$$

Podrazumeva se da je $\hat{A} = \hat{A}(\lambda)$ i $\hat{B} = \hat{B}(\lambda)$ i da je $\hat{A}\hat{A}^{-1} = \hat{I}$ $(\hat{A}^{-1}\hat{A} = \hat{I})$

$$\frac{d}{d\lambda}(\hat{A}\hat{A}^{-1}) = \frac{d\hat{I}}{d\lambda} = \hat{O}$$

$$\frac{d\hat{A}}{d\lambda}\hat{A}^{-1} + \hat{A}\frac{d\hat{A}^{-1}}{d\lambda} = \hat{O}$$

$$^{-1}/ \quad \frac{d\hat{A}}{d\lambda}\hat{A}^{-1} = -\hat{A}\frac{d\hat{A}^{-1}}{d\lambda}$$

$$\hat{A}^{-1}\frac{d\hat{A}}{d\lambda}\hat{A}^{-1} = -\underbrace{\hat{A}^{-1}\hat{A}}_{\hat{I}}\frac{d\hat{A}^{-1}}{d\lambda} \Rightarrow$$

$$\frac{d\hat{A}^{-1}}{d\lambda} = -\hat{A}^{-1}\frac{d\hat{A}}{d\lambda}\hat{A}^{-1}$$

Ako se ovde smeni $\hat{A} \to \hat{A}^{-1}$ i iskoristi $(\hat{A}^{-1})^{-1} = \hat{A}$

$$\frac{d\hat{A}}{d\lambda} = -\hat{A}\frac{d\hat{A}^{-1}}{d\lambda}\hat{A}$$

13. Data je analiticka operatorsna f-ja $f(\hat{A})$. Dokazati da je njena svojstvena jednanost data izrazom

$$f(\hat{A})\,|a\rangle = f(a)\,|a\rangle$$

gde su a i $|a\rangle$, redom: svojstvene vrednosti i svojstvena stanja opservable $\hat{A}$.

Dakle

$$\hat{A}\,|a\rangle = a\,|a\rangle \quad \Rightarrow \quad \hat{A}^n\,|a\rangle = a^n\,|a\rangle$$

Analiticka f-ja $f(\hat{A}) \Rightarrow$

$$f(\hat{A}) = \sum_n c_n \hat{A}^n$$

Zato

$$f(\hat{A})\,|a\rangle = \sum_n c_n \hat{A}^n\,|a\rangle = \sum_n c_n a^n\,|a\rangle$$

$$= f(a)\,|a\rangle$$

$$e^{\hat{A}}\,|a\rangle = e^{a}\,|a\rangle$$

$$[e^{\hat{A}}, \hat{A}]\,|a\rangle = 0\,|a\rangle \Rightarrow [e^{\hat{A}}, \hat{A}] = 0$$

$\downarrow$

Koristi se kod BHC teoreme!

3. Ako su $\hat{A}$ i $\hat{B}$ nekomutativni operatori a η parametar, tada je:

$$e^{\eta\hat{A}}\hat{B}e^{-\eta\hat{A}} = \sum_{k=0}^{\infty} \frac{\eta^k}{k!}[\hat{A},[\hat{A},\dots[\hat{A},\hat{B}]],\dots]$$

gde se podrazumeva da sa desne strane ima k srednjih (komutatorskih) zagrada. Dokazati.

$$\sum_{k=0}^{\infty} \frac{\eta^k}{k!}[\hat{A},[\hat{A},\dots[\hat{A},\hat{B}]],\dots] =$$

$$\hat{B} + \eta[\hat{A},\hat{B}] + \frac{\eta^2}{2}[\hat{A},[\hat{A},\hat{B}]] + \frac{\eta^3}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] + \dots$$

Definisimo f-ju

$$f(\eta) = e^{\eta\hat{A}}\hat{B}e^{-\eta\hat{A}}, \quad f(0) = \hat{B}$$

Razvijmo f-ju $f(\eta)$ u Maclaurin-ov red

$$f(\eta) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\eta^k \qquad (*)$$

$$f^{(k)}(0) = \frac{d^k f}{d\eta^k}\Big|_{\eta=0}$$

$$\frac{df(\eta)}{d\eta} = \frac{de^{\eta\hat{A}}}{d\eta}\hat{B}e^{-\eta\hat{A}} + e^{\eta\hat{A}}\hat{B}\frac{de^{-\eta\hat{A}}}{d\eta}$$

$$= \hat{A}e^{\eta\hat{A}}\hat{B}e^{-\eta\hat{A}} + e^{\eta\hat{A}}\hat{B}(-\hat{A})e^{-\eta\hat{A}}$$

$$= e^{\eta\hat{A}}\hat{A}\hat{B}e^{-\eta\hat{A}} - e^{\eta\hat{A}}\hat{B}\hat{A}e^{-\eta\hat{A}}$$

$$= e^{\eta\hat{A}}[\hat{A},\hat{B}]e^{-\eta\hat{A}}$$

$$\frac{d^2 f}{d\eta^2} = \frac{d}{d\eta}\left(e^{\eta \hat{A}} [\hat{A}, \hat{B}] e^{-\eta\hat{A}}\right)$$

$$= \frac{de^{\eta\hat{A}}}{d\eta}[\hat{A},\hat{B}] e^{-\eta\hat{A}} + e^{\eta\hat{A}}[\hat{A},\hat{B}] \frac{de^{-\eta\hat{A}}}{d\eta}$$

$$= A e^{\eta\hat{A}}[\hat{A},\hat{B}] e^{-\eta\hat{A}} + e^{\eta\hat{A}}[\hat{A},\hat{B}](-\hat{A}) e^{-\eta\hat{A}}$$

$$= e^{\eta\hat{A}}\hat{A}[\hat{A},\hat{B}] e^{-\eta\hat{A}} - e^{\eta\hat{A}}[\hat{A},\hat{B}]\hat{A} e^{-\eta\hat{A}}$$

$$= e^{\eta\hat{A}}[\hat{A},[\hat{A},\hat{B}]] e^{-\eta\hat{A}}$$

Dakle,

$$\frac{df}{d\eta} = e^{\eta\hat{A}}[\hat{A},\hat{B}] e^{-\eta\hat{A}}$$

$$\frac{d^2 f}{d\eta^2} = e^{\eta\hat{A}}[\hat{A},[\hat{A},\hat{B}]] e^{-\eta\hat{A}}$$

pa će biti

$$\left.\frac{df}{d\eta}\right|_{\eta=0} = \underline{[\hat{A},\hat{B}]}$$

$$\left.\frac{d^2 f}{d\eta^2}\right|_{\eta=0} = \underline{[\hat{A},[\hat{A},\hat{B}]]}$$

Zamenom poslednjih jednakosti u (*) vidi se da važi

$$f(\eta) = e^{\eta\hat{A}}\hat{B}e^{-\eta\hat{A}} = \hat{B} + \eta[\hat{A},\hat{B}] + \frac{\eta^2}{2}[\hat{A},[\hat{A},\hat{B}]] + \dots$$

čime je pokazano važenje relacije iz zadatka.

Dokazati važenje sledećeg operatorskog izraza

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-[\hat{A},\hat{B}]/2} = e^{\hat{B}} e^{\hat{A}} e^{[\hat{A},\hat{B}]/2}$$

pod uslovom $[\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0$

(uslov $[\hat{A},\hat{B}] \neq 0$)

Razmatramo f-ju oblika

$$f(\eta) = e^{\eta\hat{A}} e^{\eta\hat{B}} \quad \text{gde je } \eta \text{ parametar.}$$

Diferenciranjem po parametru

$$\frac{df(\eta)}{d\eta} = \hat{A} e^{\eta\hat{A}} e^{\eta\hat{B}} + e^{\eta\hat{A}} \hat{B} e^{\eta\hat{B}}$$

$$= \hat{A} e^{\eta\hat{A}} e^{\eta\hat{B}} + e^{\eta\hat{A}} \hat{B} \underbrace{e^{-\eta\hat{A}} e^{\eta\hat{A}}}_{I} e^{\eta\hat{B}}$$

$$= (\hat{A} + e^{\eta\hat{A}} \hat{B} e^{-\eta\hat{A}}) e^{\eta\hat{A}} e^{\eta\hat{B}}$$

$$= (\hat{A} + \hat{B} + \eta[\hat{A},\hat{B}] + \frac{\eta^2}{2}[\hat{A},[\hat{A},\hat{B}]] + \dots) f(\eta)$$

$$= (\hat{A} + \hat{B} + \eta[\hat{A},\hat{B}]) f(\eta)$$

Dakle,

$$\frac{df(\eta)}{d\eta} = ((\hat{A}+\hat{B}) + \eta[\hat{A},\hat{B}]) f(\eta)$$

Formalnom integracijom poslednjeg izraza

$$\int \frac{df}{f} = (\hat{A}+\hat{B}) \int d\eta + [\hat{A},\hat{B}] \int \eta \, d\eta + C$$

$$\ln f(\eta) = (\hat{A}+\hat{B}) \eta + \frac{[\hat{A},\hat{B}] \eta^2}{2} + C$$

$$f(0) = I \Rightarrow C = 0$$

tu $f(\eta) = (\hat{A}+\hat{B})\eta + \frac{[\hat{A},\hat{B}]}{2}\eta^2$

$$f(\eta) = e^{(\hat{A}+\hat{B})\eta + \frac{[\hat{A},\hat{B}]}{2}\eta^2}$$ odnosno

$$e^{\eta\hat{A}}e^{\eta\hat{B}} = e^{(\hat{A}+\hat{B})\eta + \frac{[\hat{A},\hat{B}]}{2}\eta^2}$$

Za $\eta = 1$

$$e^{\hat{A}}e^{\hat{B}} = e^{(\hat{A}+\hat{B}) + \frac{[\hat{A},\hat{B}]}{2}} \quad / \, e^{-[\hat{A},\hat{B}]/2}$$

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-[\hat{A},\hat{B}]/2}$$

Druga jednakost se dobija kada se krene od f-je oblika

$$f(\eta) = e^{\eta\hat{B}}e^{\eta\hat{A}}$$ Domaći

5. Za 1D sistem $\{x, p_x\}$, za čije opservable važi $[\hat{X}, \hat{P}_x] = i\hbar \hat{I}$, gde je $\hat{I}$ jedinični operator, dokazati da za proizvoljnu opservablu $\hat{A} = \hat{A}(\hat{X}, \hat{P}_x)$ važe sledeće komutacione relacije:

i) $[\hat{X}, \hat{A}] = i\hbar \dfrac{\partial \hat{A}}{\partial \hat{P}_x}$

ii) $[\hat{P}_x, \hat{A}] = -i\hbar \dfrac{\partial \hat{A}}{\partial \hat{X}}$

Najopštiji zapis opservable za 1D sistem

$$\hat{A} = \hat{A}(\hat{X}, \hat{P}) = \sum_{m,n} \alpha_{mn} \hat{X}^m \hat{P}_x^n + \beta_{mn} \hat{P}_x^m \hat{X}^n$$

Nadalje, operatori bez kapica.

i) $$[\hat{X}, \hat{A}] = [\hat{X}, \sum_{m,n} \alpha_{mn} \hat{X}^m \hat{P}_x^n + \beta_{mn} \hat{P}_x^m \hat{X}^n]$$

$$= \sum_{m,n} [\hat{X}, \alpha_{mn} \hat{X}^m \hat{P}_x^n + \beta_{mn} \hat{P}_x^m \hat{X}^n]$$

$$= \sum_{m,n} \alpha_{mn} [\hat{X}, \hat{X}^m \hat{P}_x^n] + \beta_{mn} [\hat{X}, \hat{P}_x^m \hat{X}^n]$$

$$= \sum_{m,n} \alpha_{mn} \left(\cancelto{0}{[\hat{X}, \hat{X}^m]} \hat{P}_x^n + \hat{X}^m [\hat{X}, \hat{P}_x^n] \right) + \beta_{mn} \left([\hat{X}, \hat{P}_x^m] \hat{X}^n + \hat{P}_x^m \cancelto{0}{[\hat{X}, \hat{X}^n]} \right)$$

$$= \sum_{m,n} \alpha_{mn} \hat{X}^m [\hat{X}, \hat{P}_x^n] + \beta_{mn} [\hat{X}, \hat{P}_x^m] \hat{X}^n$$

$$= \sum_{m,n} \alpha_{mn} \hat{X}^m \left(\sum_{s=0}^{n-1} \hat{P}_x^s [\hat{X}, \hat{P}_x] \hat{P}_x^{n-s-1} \right) +$$
$$\beta_{mn} \left(\sum_{s=0}^{m-1} \hat{P}_x^s [\hat{X}, \hat{P}_x] \hat{P}_x^{m-s-1} \right) \hat{X}^n$$

$$= \sum_{m,n} i\hbar \alpha_{mn} \hat{X}^m \hat{P}_x^{n-1} \underbrace{\sum_{s=0}^{n-1} 1}_{n} + i\hbar \beta_{mn} \hat{P}_x^{m-1} \underbrace{\sum_{s=0}^{m-1} 1}_{m} \hat{X}^n$$

$$= i\hbar \sum_{m,n} \alpha_{mn} \hat{X}^m n \hat{P}_x^{n-1} + \beta_{mn} m \hat{P}_x^{m-1} \hat{X}^n$$

$$= i\hbar \frac{\partial \hat{A}(x, P_x)}{\partial \hat{P}_x}$$

i) $[\hat{P}_x, \hat{A}] = [\hat{P}_x, \sum_{m,n} \alpha_{mn} \hat{X}^m \hat{P}_x^n + \beta_{mn} \hat{P}_x^m \hat{X}^n]$

$$= \sum_{m,n} \alpha_{mn} [\hat{P}_x, \hat{X}^m \hat{P}_x^n] + \beta_{mn} [\hat{P}_x, \hat{P}_x^m \hat{X}^n]$$

$$= \sum_{m,n} \alpha_{mn} \left([\hat{P}_x, \hat{X}^m] \hat{P}_x^n + \hat{X}^m \cancel{[\hat{P}_x, \hat{P}_x^n]}^{0} \right) +$$
$$\beta_{mn} \left(\cancel{[\hat{P}_x, \hat{P}_x^m]}^{0} \hat{X}^n + \hat{P}_x^m [\hat{P}_x, \hat{X}^n] \right)$$

$$= \sum_{m,n} \alpha_{mn} \sum_{s=0}^{m-1} \hat{X}^s [\hat{P}_x, \hat{X}] X^{m-s-1} \hat{P}_x^n +$$
$$\hat{P}_x^m \beta_{mn} \sum_{s=0}^{n-1} \hat{X}^s [\hat{P}_x, \hat{X}] \hat{X}^{n-s-1} =$$

$$= -i\hbar \sum_{m,n} \alpha_{mn} \hat{X}^{m-1} \sum_{s=0}^{m-1} 1 \, \hat{P}_x^n +$$
$$\beta_{mn} \hat{P}_x^m \hat{X}^{n-1} \sum_{s=0}^{n-1} 1$$

$$= -i\hbar \sum_{m,n} \alpha_{mn}\, m\hat{x}^{m-1}\hat{p}_x^{\,n} + \beta_{mn}\, \hat{p}_x^{\,m}\, n\hat{x}^{n-1}$$

$$= -i\hbar \frac{\partial \hat{A}(x, p_x)}{\partial \hat{x}}$$

6. Data je vektorska opservabla $\vec{\hat{\sigma}}$, čije deka-rtovske komponente zadovoljavaju antikomutacioni izraz

$$\{\hat{\sigma}_i, \hat{\sigma}_j\} = 2\delta_{ij}$$

komutacioni izraz

$$[\hat{\sigma}_i, \hat{\sigma}_j] = 2i\,\varepsilon_{ijk}\,\hat{\sigma}_k$$

Dokazati da važi:

$$(\vec{\hat{\sigma}}\cdot\vec{A})(\vec{\hat{\sigma}}\cdot\vec{B}) = \vec{A}\cdot\vec{B} + i\,\vec{\hat{\sigma}}(\vec{A}\times\vec{B})$$

gde su $\vec{A}$ i $\vec{B}$ obični 3D ~~v~~ektori a ε_{ijk} je simbol Levi-Čivita.

$$(\vec{\hat{\sigma}}\cdot\vec{A})(\vec{\hat{\sigma}}\cdot\vec{B}) = \hat{\sigma}_i A_i\,\hat{\sigma}_j B_j = \hat{\sigma}_i\hat{\sigma}_j A_i B_j \qquad (*)$$

Iz komutatora i antikomutatora $([\,]+\{\,\}) \Rightarrow$

$$\hat{\sigma}_i\hat{\sigma}_j = \delta_{ij} + i\,\varepsilon_{ijk}\hat{\sigma}_k$$

Zamenimo to u DS $(*)$, pa će biti

$$(\vec{\hat{\sigma}}\cdot\vec{A})(\vec{\hat{\sigma}}\cdot\vec{B}) = (\delta_{ij} + i\,\varepsilon_{ijk}\hat{\sigma}_k)\,A_i B_j$$

$$= \delta_{ij}A_iB_j + i\,\varepsilon_{ijk}\hat{\sigma}_k\,A_iB_j$$

$$\vec{A}\times\vec{B} = \varepsilon_{ijk}A_iB_j\vec{e}_k$$

Def. vektorskog proizvoda dva 3D vektora

$$\cancel{(\vec{\hat{\sigma}}\cdot\vec{A})(\vec{\hat{\sigma}}\cdot\vec{B}) = A_iB_i + i\,\varepsilon_{ijk}\hat{\sigma}_k\vec{e}_k\,A_iB_j}$$

$$\cancel{= \vec{A}\cdot\vec{B} + i\,\varepsilon_{ijk}\hat{\sigma}_k\vec{e}_k\,A_iB_j\vec{e}_k}$$

Iz definicije vektorskog proizvoda

$$(\vec{A} \times \vec{B})_k = \varepsilon_{ijk} A_i B_j$$

Dakle,

$$(\vec{\sigma} \cdot \vec{A})(\vec{\sigma} \cdot \vec{B}) = \vec{A} \cdot \vec{B} + i\, \hat{\sigma}_k (\vec{A} \times \vec{B})_k$$

$$= \vec{A} \cdot \vec{B} + i\, \hat{\sigma}_k (\vec{A} \times \vec{B})_k$$

$$= \vec{A} \cdot \vec{B} + i\, \hat{\vec{\sigma}} (\vec{A} \times \vec{B})$$

7. U Hilbertovom prostoru $\mathcal{H}^{(u)}$ koji predstavlja tenzorski (direktni) proizvod dva Hilbertova prostora $\mathcal{H}^{(u)} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$, date su opservabl. $\hat{A}_1, \hat{B}_1, \hat{A}_2, \hat{B}_2$. Dokazati jednakost:

$$[\hat{A}_1 \otimes \hat{A}_2, \hat{B}_1 \otimes \hat{B}_2] = [\hat{A}_1, \hat{B}_1] \otimes \hat{A}_2 \hat{B}_2 + \hat{B}_1 \hat{A}_1 \otimes [\hat{A}_2, \hat{B}_2]$$

a potom je uopštiti na slučaj kada su u pitanju tri faktor prostora $\mathcal{H}^{(u)} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \otimes \mathcal{H}^{(3)}$ sa odgovarajućim opservablama.

$$\mathcal{H}^{(u)} = \underset{(\hat{A}_1, \hat{B}_1)}{\mathcal{H}^{(1)}} \otimes \underset{(\hat{A}_2, \hat{B}_2)}{\mathcal{H}^{(2)}}$$

$$[\hat{A}_1 \otimes \hat{A}_2, \hat{B}_1 \otimes \hat{B}_2] \overset{d}{=} (\hat{A}_1 \otimes \hat{A}_2)(\hat{B}_1 \otimes \hat{B}_2) - (\hat{B}_1 \otimes \hat{B}_2)(\hat{A}_1 \otimes \hat{A}_2)$$

$$= \hat{A}_1 \hat{B}_1 \otimes \hat{A}_2 \hat{B}_2 - \hat{B}_1 \hat{A}_1 \otimes \hat{B}_2 \hat{A}_2 + \hat{B}_1 \hat{A}_1 \otimes \hat{A}_2 \hat{B}_2 - \hat{B}_1 \hat{A}_1 \otimes \hat{A}_2 \hat{B}_2$$

$$= [\hat{A}_1, \hat{B}_1] \otimes \hat{A}_2 \hat{B}_2 + \hat{B}_1 \hat{A}_1 \otimes [\hat{A}_2, \hat{B}_2]$$

Kada su pitanju tri faktor prostora

$$\mathcal{H}^{(u)} = \underset{(\hat{A}_1, \hat{B}_1)}{\mathcal{H}^{(1)}} \otimes \underset{(\hat{A}_2, \hat{B}_2)}{\mathcal{H}^{(2)}} \otimes \underset{(\hat{A}_3, \hat{B}_3)}{\mathcal{H}^{(3)}}$$

Treba izračunati komutator, gde se koristi prethodni rezultat

$$[\hat{A}_1 \otimes \underbrace{\hat{A}_2 \otimes \hat{A}_3}_{\hat{\varphi}_2}, \hat{B}_1 \otimes \underbrace{\hat{B}_2 \otimes \hat{B}_3}_{\hat{\Psi}_2}] = [\hat{A}_1 \otimes \hat{\varphi}_2, \hat{B}_1 \otimes \hat{\Psi}_2]$$

$$= [\hat{A}_1, \hat{B}_1] \otimes \hat{\varphi}_2 \hat{\psi}_2 + \hat{B}_1 \hat{A}_1 \otimes [\hat{\varphi}_2, \hat{\psi}_2] =$$

$$= [\hat{A}_1, \hat{B}_1] \otimes (\hat{A}_2 \otimes \hat{A}_3)(\hat{B}_2 \otimes \hat{B}_3) + \hat{B}_1 \hat{A}_1 \otimes [\hat{A}_2 \otimes \hat{A}_3, \hat{B}_2 \otimes \hat{B}_3]$$

$$= [\hat{A}_1, \hat{B}_1] \otimes \hat{A}_2 \hat{B}_2 \otimes \hat{A}_3 \hat{B}_3 + \hat{B}_1 \hat{A}_1 \otimes \left([\hat{A}_2, \hat{B}_2] \otimes \hat{A}_3 \hat{B}_3 + \hat{B}_2 \hat{A}_2 \otimes [\hat{A}_3, \hat{B}_3] \right) =$$

$$= [\hat{A}_1, \hat{B}_1] \otimes \hat{A}_2 \hat{B}_2 \otimes \hat{A}_3 \hat{B}_3 + \hat{B}_1 \hat{A}_1 \otimes [\hat{A}_2, \hat{B}_2] \otimes \hat{A}_3 \hat{B}_3 + \hat{B}_1 \hat{A}_1 \otimes \hat{B}_2 \hat{A}_2 \otimes [\hat{A}_3, \hat{B}_3]$$

Kako će glasiti komutator ako su u pitanju 4 faktor prostora?

$$[\hat{A}_1 \otimes \hat{A}_2 \otimes \hat{A}_3 \otimes \hat{A}_4, \hat{B}_1 \otimes \hat{B}_2 \otimes \hat{B}_3 \otimes \hat{B}_4] =$$

$$[\hat{A}_1, \hat{B}_1] \otimes \hat{A}_2 \hat{B}_2 \otimes \hat{A}_3 \hat{B}_3 \otimes \hat{A}_4 \hat{B}_4 +$$

$$\hat{B}_1 \hat{A}_1 \otimes [\hat{A}_2, \hat{B}_2] \otimes \hat{A}_3 \hat{B}_3 \otimes \hat{A}_4 \hat{B}_4 +$$

$$\hat{B}_1 \hat{A}_1 \otimes \hat{B}_2 \hat{A}_2 \otimes [\hat{A}_3, \hat{B}_3] \otimes \hat{A}_4 \hat{B}_4 +$$

$$\hat{B}_1 \hat{A}_1 \otimes \hat{B}_2 \hat{A}_2 \otimes \hat{B}_3 \hat{A}_3 \otimes [\hat{A}_4, \hat{B}_4]$$

8. Na osnovi rezultata prethodnog zadatka eksplicitnim računom proveriti komutacione relacije komponenata vektora položaja i impulsa, tj. dokazati

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}, \quad [\hat{x}_i, \hat{x}_j] = 0 = [\hat{p}_i, \hat{p}_j], \forall i,$$

Vektorske opservable (konjugovane)

$$\hat{\vec{r}} = (\hat{x}, \hat{y}, \hat{z}) \equiv (\hat{x}_1, \hat{x}_2, \hat{x}_3)$$

$$\hat{\vec{p}} = (\hat{p}_x, \hat{p}_y, \hat{p}_z) \equiv (\hat{p}_1, \hat{p}_2, \hat{p}_3)$$

$$\hat{x}_1 = \hat{x}_1 \otimes \hat{I}_2 \otimes \hat{I}_3 \qquad \hat{p}_1 = \hat{p}_1 \otimes \hat{I}_2 \otimes \hat{I}_3$$

$$\hat{x}_2 = \hat{I}_1 \otimes \hat{x}_2 \otimes \hat{I}_3 \qquad \hat{p}_2 = \hat{I}_1 \otimes \hat{p}_2 \otimes \hat{I}_3$$

$$\hat{x}_3 = \hat{I}_1 \otimes \hat{I}_2 \otimes \hat{x}_3 \qquad \hat{p}_3 = \hat{I}_1 \otimes \hat{I}_2 \otimes \hat{p}_3$$

$$[\hat{x}_1, \hat{x}_2] = [\hat{x}_1 \otimes \hat{I}_2 \otimes \hat{I}_3, \hat{I}_1 \otimes \hat{x}_2 \otimes \hat{I}_3]$$

$$= \cancel{[x_1, \hat{I}_1]}^{0} \otimes \hat{I}_2\hat{x}_2 \otimes \hat{I}_3\hat{I}_3 + \hat{I}_1\hat{x}_1 \otimes \cancel{[\hat{I}_2, \hat{x}_2]}^{0} \otimes \hat{I}_3\hat{I}_3$$

$$+ \hat{I}_1\hat{x}_1 \otimes \hat{x}_2\hat{I}_2 \otimes \cancel{[\hat{I}_3, \hat{I}_3]}^{0} = 0$$

Isto ovako i kombinacije

$[\hat{x}_1, \hat{x}_3]$, $[\hat{x}_2, \hat{x}_3]$ a jasno je da je

$$[\hat{x}_1, \hat{x}_1] = [\hat{x}_2, \hat{x}_2] = [\hat{x}_3, \hat{x}_3] = 0 \quad i$$

$$[\hat{x}_3, \hat{x}_1] = -[\hat{x}_1, \hat{x}_3], \quad [\hat{x}_3, \hat{x}_2] = -[\hat{x}_2, \hat{x}_3]$$

$$[\hat{P}_1,\hat{P}_2] = [\hat{P}_1 \otimes \hat{I}_2 \otimes \hat{I}_3 , \hat{I}_1 \otimes \hat{P}_2 \otimes \hat{I}_3] =$$

$$= \cancel{[\hat{P}_1,\hat{I}_1]}^{0} \otimes \hat{I}_2\hat{P}_2 \otimes \hat{I}_3\hat{I}_3 + \hat{I}_1\hat{P}_1 \otimes \cancel{[\hat{I}_2,\hat{P}_2]}^{0} \otimes \hat{I}_3\hat{I}_3$$

$$+ \hat{I}_1\hat{P}_1 \otimes \hat{P}_2\hat{I}_2 \otimes \cancel{[\hat{I}_3,\hat{I}_3]}^{0} = 0$$

Isto ovako i kombinacije

$[\hat{P}_1,\hat{P}_3]$, $[\hat{P}_2,\hat{P}_3]$ a jasno je da je

$[\hat{P}_1,\hat{P}_1] = [\hat{P}_2,\hat{P}_2] = [\hat{P}_3,\hat{P}_3] = 0$ i

$[\hat{P}_3,\hat{P}_1] = -[\hat{P}_1,\hat{P}_3]$, $[\hat{P}_3,\hat{P}_2] = -[\hat{P}_2,\hat{P}_3]$

$$[\hat{X}_1,\hat{P}_2] = [\hat{X}_1 \otimes \hat{I}_2 \otimes \hat{I}_3 , \hat{I}_1 \otimes \hat{P}_2 \otimes \hat{I}_3]$$

$$= \cancel{[\hat{X}_1,\hat{I}_1]}^{0} \otimes \hat{I}_2\hat{P}_2 \otimes \hat{I}_3\hat{I}_3 + \hat{I}_1\hat{X}_1 \otimes \cancel{[\hat{I}_2,\hat{P}_2]}^{0} \otimes \hat{I}_3\hat{I}_3$$

$$+ \hat{I}_1\hat{X}_1 \otimes \hat{P}_2\hat{I}_2 \otimes \cancel{[\hat{I}_3,\hat{I}_3]}^{0} = 0$$

$$[\hat{X}_1,\hat{P}_1] = [\hat{X}_1 \otimes \hat{I}_2 \otimes \hat{I}_3 , \hat{P}_1 \otimes \hat{I}_2 \otimes \hat{I}_3] =$$

$$= [\hat{X}_1,\hat{P}_1] \otimes \hat{I}_2\hat{I}_2 \otimes \hat{I}_3\hat{I}_3 + \hat{P}_1\hat{X}_1 \otimes \cancel{[\hat{I}_2,\hat{I}_2]}^{0} \otimes \hat{I}_3\hat{I}_3$$

$$+ \hat{P}_1\hat{X}_1 \otimes \hat{I}_2\hat{I}_2 \otimes \cancel{[\hat{I}_3,\hat{I}_3]}^{0}$$

$$= [\hat{X},\hat{P}_x] = i\hbar$$

Slično se pokaže i $[\hat{X}_1,\hat{P}_3]$, $[\hat{X}_2,\hat{P}_3]$ i

$[\hat{X}_2,\hat{P}_2]$ i $[\hat{X}_3,\hat{P}_3]$

12. Za trodimenzionalnu česticu čije opservable položaja i impulsa zadovoljavaju komutacione relacije $[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}$, $[\hat{x}_i, \hat{x}_j] = 0$, $[\hat{p}_i, \hat{p}_j] = 0$ i-ta komponenta operatora momenta impulsa definisana je izrazom

$$\hat{L}_i = \varepsilon_{ijk} \hat{x}_j \hat{p}_k$$

Pokazati da komponente operatora $\hat{\vec{L}}$ zadovoljavaju komutacione izraze

$$[\hat{L}_i, \hat{L}_j] = i\hbar \epsilon_{ijk} \hat{L}_k$$

gde je ϵ_{ijk} simbol Levi-Čivita.

$$\hat{\vec{L}} \stackrel{d}{=} \hat{\vec{r}} \times \hat{\vec{p}} = \varepsilon_{ijk} \vec{e}_i \hat{x}_j \hat{p}_k$$

$$\hat{L}_i = \varepsilon_{ipq} \hat{x}_p \hat{p}_q$$

$$\hat{L}_j = \varepsilon_{jmn} \hat{X}_m \hat{p}_n$$

$$[\hat{L}_i, \hat{L}_j] = [\varepsilon_{ipq} \hat{x}_p \hat{p}_q, \varepsilon_{jmn} \hat{X}_m \hat{p}_n] =$$

$$= \sum_{p,q} \sum_{m,n} \varepsilon_{ipq} \varepsilon_{jmn} [\hat{X}_p \hat{P}_q, \hat{X}_m \hat{P}_n]$$

$$= \sum_{p,q} \sum_{m,n} \varepsilon_{ipq} \varepsilon_{jmn} \left([\hat{X}_p \hat{P}_q, \hat{X}_m] \hat{P}_n + \hat{X}_m [\hat{X}_p \hat{P}_q, \hat{P}_n] \right)$$

$$= \sum_{p,q} \sum_{m,n} \varepsilon_{ipq} \varepsilon_{jmn} \left(\{ \overset{0}{\cancel{[\hat{X}_p, \hat{X}_m]}} \hat{P}_q + \hat{X}_p [\hat{P}_q, \hat{X}_m] \} \hat{P}_n + \right.$$

$$\left. + \hat{X}_m \{ [\hat{X}_p, \hat{P}_n] \hat{P}_q + \hat{X}_p \overset{0}{\cancel{[\hat{P}_q, \hat{P}_n]}} \} \right)$$

$$= \sum_{p,q} \sum_{m,n} \varepsilon_{ipq}\, \varepsilon_{jmn} \left(-i\hbar\, \delta_{qm}\, \hat{x}_p \hat{p}_n + i\hbar\, \delta_{pn}\, \hat{x}_m \hat{p}_q \right)$$

$$= i\hbar \sum_{p,q} \sum_{m,n} \left(-\epsilon_{ipq}\, \epsilon_{jmn}\, \delta_{qm}\, \hat{x}_p \hat{p}_n + \varepsilon_{ipq}\, \varepsilon_{jmn}\, \delta_{pn}\, \hat{x}_m \hat{p}_q \right)$$

$$= i\hbar \left(\sum_{p} \sum_{m,n} -\epsilon_{ipm}\, \epsilon_{jmn}\, \hat{x}_p \hat{p}_n + \sum_{q} \sum_{m,n} \epsilon_{inq}\, \epsilon_{jmn}\, \hat{x}_m \hat{p}_q \right)$$

$$= i\hbar \left(\sum_{q,m} \sum_{n} \epsilon_{inq}\, \epsilon_{jmn}\, \hat{x}_m \hat{p}_q - \sum_{p,n} \sum_{m} \epsilon_{ipm}\, \epsilon_{jmn}\, \hat{x}_p \hat{p}_n \right)$$

Veza Levi-Civita simbola i Kronekerove delte

СУМИРАЊЕ

$$\sum_i \epsilon_{ijk}\, \epsilon_{ilm} = \delta_{jl}\, \delta_{km} - \delta_{jm}\, \delta_{kl}$$

$$\begin{array}{c|c} j & k \\ \hline l & m \end{array} \; - \; \begin{array}{c|c} j & k \\ \hline l & m \end{array}$$

Ali, pre primene veze ϵ i δ treba ϵ izraze (tj. njihove indekse) u poslednjoj jednakosti preurediti

$$\sum_n \epsilon_{inq}\, \epsilon_{jmn} = \epsilon_{inq}\, \epsilon_{jmn} \overset{(*)}{=} -\epsilon_{niq}\, \epsilon_{njm}$$

$$\epsilon_{inq} = -\epsilon_{niq}$$

$$\epsilon_{jmn} = -\epsilon_{jnm} = \epsilon_{njm} \quad (*)$$

$$\sum_m \epsilon_{ipm}\epsilon_{jmn} = \epsilon_{ipm}\epsilon_{jmn} \overset{(**)}{=} -\epsilon_{mjn}\epsilon_{mip}$$

$$\epsilon_{ipm} = -\epsilon_{imp} = \epsilon_{mip}$$
$$\epsilon_{jmn} = -\epsilon_{mjn} \qquad (**)$$

$$= i\hbar\Big(-\sum_{q,m}\epsilon_{niq}\epsilon_{njm}\hat{X}_m\hat{P}_q + \sum_{p,n}\epsilon_{mjn}\epsilon_{mip}\hat{X}_p\hat{P}_n\Big)$$

$$= i\hbar\Big(-\sum_{q,m}(\delta_{ij}\delta_{qm} - \delta_{im}\delta_{qj})\cdot\hat{X}_m\hat{P}_q + \sum_{p,n}(\delta_{ji}\delta_{np} - \delta_{ni}\delta_{jp})\cdot\hat{X}_p\hat{P}_n\Big)$$

$$= i\hbar\Big(-\sum_{q,m}\delta_{ij}\delta_{qm}\hat{X}_m\hat{P}_q + \sum_{q,m}\delta_{im}\delta_{qj}\hat{X}_m\hat{P}_q +$$
$$+\sum_{p,n}\delta_{ji}\delta_{np}\hat{X}_p\hat{P}_n - \sum_{p,n}\delta_{ni}\delta_{jp}\hat{X}_p\hat{P}_n\Big)$$

$$= i\hbar\Big(\cancel{-\delta_{ij}\sum_m \hat{X}_m\hat{P}_m} + \hat{X}_i\hat{P}_j + \cancel{\delta_{ij}\sum_n \hat{X}_n\hat{P}_n} - \hat{X}_j\hat{P}_i\Big)$$

$$= i\hbar(\hat{X}_i\hat{P}_j - \hat{X}_j\hat{P}_i) = i\hbar\sum_{m,n}(\delta_{im}\delta_{jn} - \delta_{jm}\delta_{in})\hat{X}_m\hat{P}_n$$

$$= i\hbar\sum_{s,m,n}\epsilon_{sij}\epsilon_{smn}\hat{X}_m\hat{P}_n \qquad \begin{vmatrix} i & j \\ m & n \end{vmatrix} - \begin{matrix} i & j \\ m & n \end{matrix}$$

$$= i\hbar\sum_s \epsilon_{sij}\sum_{m,n}\epsilon_{smn}\hat{X}_m\hat{P}_n = \underline{i\hbar\sum_s \epsilon_{sij}\hat{L}_s}$$

10. Za trodimenzionalni sistem definisan uz pomoć relacija $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$, $[\hat{x}_i, \hat{x}_j] = 0$ i $[\hat{p}_i, \hat{p}_j] = 0$ dokazati komutacione relacije!

$$[\hat{L}_i, \hat{x}_j] = i\hbar\epsilon_{ijk}\hat{x}_k$$

$$[\hat{p}_i, \hat{L}_j] = i\hbar\epsilon_{ijk}\hat{p}_k$$

$$\hat{L}_i = \epsilon_{ipq}\hat{x}_p\hat{p}_q$$

$$[\hat{L}_i, \hat{x}_j] = \epsilon_{ipq}[\hat{x}_p\hat{p}_q, \hat{x}_j] = \epsilon_{ipq}\left(\cancel{[\hat{x}_p, \hat{x}_j]}\hat{p}_q + \hat{x}_p[\hat{p}_q, \hat{x}_j]\right)$$

$$= \epsilon_{ipq}\hat{x}_p(-i\hbar\delta_{qj}) = -i\hbar\epsilon_{ipj}\hat{x}_p = i\hbar\epsilon_{ijp}\hat{x}_p$$

$$\hat{L}_j = \epsilon_{jmn}\hat{x}_m\hat{p}_n$$

$$[\hat{p}_i, \hat{L}_j] = \epsilon_{jmn}[\hat{p}_i, \hat{x}_m\hat{p}_n] = \epsilon_{jmn}\left(\hat{x}_m[\hat{p}_i, \hat{p}_n] + [\hat{p}_i, \hat{x}_m]\hat{p}_n\right) = -\epsilon_{jmn}i\hbar\delta_{im}\hat{p}_n = +i\hbar\epsilon_{ijn}\hat{p}_n$$

11. Pokazati važenje relacija

a) $[\hat{\vec{r}}, \hat{A}(\hat{\vec{r}},\hat{\vec{p}})] = i\hbar \frac{\partial \hat{A}}{\partial \hat{\vec{p}}}$

b) $[\hat{\vec{p}}, \hat{A}(\hat{\vec{r}},\hat{\vec{p}})] = -i\hbar \frac{\partial \hat{A}}{\partial \hat{\vec{r}}}$

$$\hat{A}(\hat{\vec{r}},\hat{\vec{p}}) = \sum_{i,j=1}^{3} \sum_{m_i,n_j} C_{m_i n_j} \hat{X}_i^{m_i} \hat{p}_j^{n_j} + d_{m_i n_j} \hat{p}_i^{m_i} \hat{X}_j^{n_j}$$

$$\hat{\vec{r}} = (\hat{X}_1, \hat{X}_2, \hat{X}_3) \equiv \{\hat{X}_r\}$$

$$[\hat{X}_r, \hat{A}] = [\hat{X}_r, \sum_{i,j}\sum_{m_i,n_j} C_{m_i n_j} \hat{X}_i^{m_i} \hat{p}_j^{n_j} + d_{m_i n_j} \hat{p}_i^{m_i} \hat{X}_j^{n_j}]$$

$$= \sum_{i,j}\sum_{m_i,n_j} [\hat{X}_r, C_{m_i n_j} \hat{X}_i^{m_i} \hat{p}_j^{n_j}] + [\hat{X}_r, d_{m_i n_j} \hat{p}_i^{m_i} \hat{X}_j^{n_j}]$$

$$= \sum_{i,j}\sum_{m_i,n_j} C_{m_i n_j} [\hat{X}_r, \hat{X}_i^{m_i} \hat{p}_j^{n_j}] + d_{m_i n_j} [\hat{X}_r, \hat{p}_i^{m_i} \hat{X}_j^{n_j}]$$

$$= \sum_{i,j}\sum_{m_i,n_j} C_{m_i n_j} \{\cancel{[\hat{X}_r, \hat{X}_i^{m_i}]}^{0} \hat{p}_j^{n_j} + \hat{X}_i^{m_i} [\hat{X}_r, \hat{p}_j^{n_j}]\} +$$

$$d_{m_i n_j} \{[\hat{X}_r, \hat{p}_i^{m_i}] \hat{X}_j^{n_j} + \hat{p}_i^{m_i} \cancel{[\hat{X}_r, \hat{X}_j^{n_j}]}^{0}\} =$$

$$\sum_{i,j}\sum_{m_i,n_j} C_{m_i n_j} \hat{X}_i^{m_i} [\hat{X}_r, \hat{p}_j^{n_j}] + d_{m_i n_j} [\hat{X}_r, \hat{p}_i^{m_i}] \hat{X}_j^{n_j}$$

$$[\hat{X}_r, \hat{p}_j^{n_j}] = i\hbar \sum_{s=0}^{n_j-1} \hat{p}_j^{s} \hat{p}_j^{n_j-s-1} \delta_{jr} = i\hbar\, \hat{p}_j^{n_j-1} \delta_{jr} \sum_{s=0}^{n_j-1} 1$$

$[X_j, P_j^{n_j}] = i\hbar\, n_j\, P_j^{n_j-1}\, \delta_{jr}$

Slično

$[\hat{X}_r, \hat{P}_i^{m_i}] = i\hbar\, m_i\, P_i^{m_i-1}\, \delta_{ir}$

Zato

$$[\hat{X}_r, \hat{A}] = \sum_{i,j} \sum_{m_i n_j} C_{m_i n_j}\, \hat{X}_i^{m_i} \left(i\hbar\, n_j\, \hat{P}_j^{n_j-1}\, \delta_{jr} \right) +$$

$$d_{m_i n_j} \left(i\hbar\, m_i\, \hat{P}_i^{m_i-1}\, \delta_{ir} \right) \hat{X}_j^{n_j} =$$

$$= \sum_{i,j} \sum_{m_i, n_j} C_{m_i n_j}\, i\hbar\, \delta_{jr}\, n_j\, \hat{X}_i^{m_i}\, \hat{P}_j^{n_j-1} + d_{m_i n_j}\, i\hbar\, \delta_{ir}\, m_i\, \hat{P}_i^{m_i-1}\, \hat{X}_j^{n_j}$$

$$i\hbar \sum_{ij} \sum_{m_i n_j} C_{m_i n_j}\, \delta_{jr}\, n_j\, X_i^{m_i}\, P_j^{n_j-1} + d_{m_i n_j}\, \delta_{ir}\, m_i\, \hat{P}_i^{m_i-1}\, \hat{X}_j^{n_j}$$

$$i\hbar \left(\sum_i \sum_{m_i n_r} C_{m_i n_r}\, n_r\, X_i^{m_i}\, P_r^{n_r-1} + \sum_j \sum_{m_r, n_j} d_{m_r n_j}\, m_r\, \hat{P}_r^{m_r-1}\, \hat{X}_j^{n_j} \right)$$

$$= i\hbar\, \frac{\partial \hat{A}}{\partial \hat{P}_r}$$

) Domaći

$[P_r, \hat{A}] = \ldots$

12. Skalarni proizvod dveju vektorskih opservabli $\vec{\hat{\sigma}}_1$, $\vec{\hat{\sigma}}_2$ koje deluju nad različitim Hilbertovim prostorima stanja $\mathcal{H}^{(1)}$ i $\mathcal{H}^{(2)}$, definiše opservablu $\hat{\sigma} = \vec{\hat{\sigma}}_1 \cdot \vec{\hat{\sigma}}_2 = \sum_i \hat{\sigma}_{1i} \otimes \hat{\sigma}_{2i}$ koja deluje nad ukupnim prostorom stanja. Uprostiti izraz $\hat{\sigma}^k$.

Komponente opservable $\vec{\hat{\sigma}}$ zadovoljavaju relacije

$$\left.\begin{aligned} \{\hat{\sigma}_i, \hat{\sigma}_j\} &= 2\delta_{ij} \\ [\hat{\sigma}_i, \hat{\sigma}_j] &= 2i\varepsilon_{ijk}\hat{\sigma}_k \end{aligned}\right\} \Rightarrow$$

$$\hat{\sigma}_i \hat{\sigma}_j = \delta_{ij}\hat{I} + i\varepsilon_{ijk}\hat{\sigma}_k$$

U $\mathcal{H}^{(1)}$ će biti

$$\hat{\sigma}_{1i}\hat{\sigma}_{1j} = \delta_{ij}\hat{I}_1 + i\varepsilon_{ijk}\hat{\sigma}_{1k}$$

u $\mathcal{H}^{(2)}$ će biti

$$\hat{\sigma}_{2i}\hat{\sigma}_{2j} = \delta_{ij}\hat{I}_2 + i\varepsilon_{ijl}\hat{\sigma}_{2l}$$

$$\hat{\sigma}^2 = \sum_i \hat{\sigma}_{1i} \otimes \hat{\sigma}_{2i} \sum_j \hat{\sigma}_{1j} \otimes \hat{\sigma}_{2j} =$$

$$= \sum_{i,j} \hat{\sigma}_{1i}\hat{\sigma}_{1j} \otimes \hat{\sigma}_{2i}\hat{\sigma}_{2j}$$

$$= \sum_{i,j} (\delta_{ij}\hat{I}_1 + i\varepsilon_{ijk}\hat{\sigma}_{1k}) \otimes (\delta_{ij}\hat{I}_2 + i\varepsilon_{ijl}\hat{\sigma}_{2l}) =$$

$$= \sum_{i,j} \delta_{ij}\hat{I}_1 \otimes \hat{I}_2 + i\delta_{ij}\varepsilon_{ijl}\hat{I}_1 \otimes \hat{\sigma}_{2l} + i\delta_{ij}\varepsilon_{ijk}\hat{\sigma}_{1k} \otimes \hat{I}_2$$

$$- \varepsilon_{ijk}\varepsilon_{ijl}\, \hat{b}_{1k} \otimes \hat{b}_{2l} =$$

$$= 3\hat{I}_U - \sum_{i,j} \varepsilon_{ijk}\varepsilon_{ijl}\, \hat{b}_{1k} \otimes \hat{b}_{2l}$$

$$\sum_{i,j} \varepsilon_{ijk}\varepsilon_{ijl} = \varepsilon_{ijk}\varepsilon_{ijl} = \delta_{jj}\delta_{kl} - \delta_{jl}\delta_{kj}$$

$$\boxed{\begin{vmatrix} j & k \\ j & l \end{vmatrix} - \begin{vmatrix} j & k \\ j & l \end{vmatrix}}$$

Mnemoničko pravilo za ε_{abc}

$$= 3\hat{I}_U - \sum_j (\delta_{jj}\delta_{kl} - \delta_{jl}\delta_{kj})\, \hat{b}_{1k} \otimes \hat{b}_{2l}$$

$$= 3 - \left(3\,\delta_{kl}\, \hat{b}_{1k} \otimes \hat{b}_{2l} - \sum_j \delta_{jl}\delta_{kj}\, \hat{b}_{1k} \otimes \hat{b}_{2l} \right)$$

$$= 3 - \left(3\, \hat{b}_{1k} \otimes \hat{b}_{2k} - \delta_{kl}\, \hat{b}_{1k} \otimes \hat{b}_{2l} \right)$$

$$= 3 - (3\hat{b} - \hat{b}) = 3 - 2\hat{b}$$

Dakle

$$\hat{b}^2 = 3 - 2\hat{b}$$

$$\hat{b}^3 = 3\hat{b} - 2\hat{b}^2 = 3\hat{b} - 2(3 - 2\hat{b}) = 3\hat{b} - 6 + 4\hat{b}$$

$$= 7\hat{b} - 6$$

$$\hat{b}^4 = \hat{b}\cdot\hat{b}^3 = \hat{b}(7\hat{b} - 6) = 7\hat{b}^2 - 6\hat{b} = 7(3 - 2\hat{b}) - 6\hat{b}$$

$$= 21 - 20\hat{b}$$

$$b^k \begin{cases} b^{2n} = (3 - 2b)^n \\ b^{2n+1} = (3 - 2b)^n\, b \end{cases}$$

Za date operatore $\hat{A}$ i $\hat{B}$ zadat je operator $\hat{D} = (\hat{A} - \lambda\hat{B})^{-1}$, $\lambda \in R$. Naći razvoj $\hat{D}$ u stepeni red po λ.

$$\hat{D} = \sum_{n=0}^{\infty} \lambda^n \hat{D}_n \qquad \hat{D}_n = ?$$

$$(\hat{A}-\lambda\hat{B}) \Big| \quad (A - \lambda B)^{-1} = \sum_n \lambda^n \hat{D}_n$$

$$\hat{I} = (\hat{A} - \lambda\hat{B})(\hat{D}_0 + \lambda \hat{D}_1 + \lambda^2 \hat{D}_2 + \lambda^3 \hat{D}_3 + \dots)$$

$$\hat{I} = \hat{A}\hat{D}_0 + \lambda \hat{A}\hat{D}_1 + \lambda^2 \hat{A} \hat{D}_2 + \lambda^3 \hat{A}\hat{D}_3 + \dots$$
$$- \lambda \hat{B}\hat{D}_0 - \lambda^2 \hat{B}\hat{D}_1 - \lambda^3 B \hat{D}_2 - \dots$$

$\cancel{\hat{I} = (\hat{A} - \lambda\hat{B})\hat{D}_0}$ Grupisanjem članova uz $\lambda, \lambda^1, \lambda^2 \dots$

$$\hat{I} = \hat{A}\hat{D}_0 + \lambda(\hat{A}\hat{D}_1 - \hat{B}\hat{D}_0) + \lambda^2(\hat{A}\hat{D}_2 - B\hat{D}_1) +$$
$$+ \lambda^3(\hat{A}\hat{D}_3 - B\hat{D}_2) + \dots$$

$\Downarrow$

B. Komentar

$$\hat{I} = \hat{A}\hat{D}_0 \Rightarrow \hat{D}_0 = \hat{A}^{-1}$$

$$A\hat{D}_1 = B\hat{D}_0 \Rightarrow \hat{D}_1 = A^{-1} B \hat{D}_0 = A^{-1} B A^{-1}$$

$$A\hat{D}_2 = \hat{B}\hat{D}_1 \Rightarrow \hat{D}_2 = A^{-1}\hat{B}\hat{D}_1 = \hat{A}^{-1}\hat{B}(\hat{A}^{-1}\hat{B}\hat{A}^{-1})$$
$$= (\hat{A}^{-1}\hat{B})^2 \hat{A}^{-1}$$

$$A\hat{D}_3 = \hat{B}\hat{D}_2 \Rightarrow \hat{D}_3 = \hat{A}^{-1}\hat{B}\hat{D}_2 = \hat{A}^{-1}B(\hat{A}^{-1}B)^2\hat{A}^{-1}$$
$$= (\hat{A}^{-1}\hat{B})^3 \hat{A}^{-1}$$

$$\hat{D}_n = (\hat{A}^{-1}\hat{B})^n \hat{A}^{-1} \Rightarrow \hat{D} = \sum_n \lambda^n (\hat{A}^{-1}\hat{B})^n \hat{A}^{-1}$$

15. Dat je operator $\hat{a}$ koji zadovoljava sledeću komutacionu relaciju sa svojim adjungovanim operatorom $[\hat{a}, \hat{a}^+] = \hat{I}$, gde je $\hat{I}$ identični operator. Za operator $\hat{N} = \hat{a}^+\hat{a}$ važi svojstvena jednakost $\hat{N}|n\rangle = n|n\rangle$. Naći način delovanja operatora $\hat{a}$ i $\hat{a}^+$ na stanja $|n\rangle$.

$$[\hat{a}, \hat{a}^+] = \hat{I} \quad , \quad \hat{N}|n\rangle = n|n\rangle$$

$$\hat{a}|n\rangle = ? \qquad \hat{a}^+|n\rangle = ?$$

$$\hat{a}\hat{a}^+ - \hat{a}^+\hat{a} = \hat{I}$$

$$\hat{a}\hat{a}^+ - \hat{I} = \hat{a}^+\hat{a}$$

$$\hat{N}(\hat{a}|n\rangle) \stackrel{d}{=} \hat{a}^+\hat{a}\,\hat{a}|n\rangle = (\hat{a}\hat{a}^+ - \hat{I})\,\hat{a}|n\rangle =$$

$$= \hat{a}\hat{a}^+\hat{a}|n\rangle - \hat{a}|n\rangle = \hat{a}\,n|n\rangle - \hat{a}|n\rangle = \hat{a}(n-1)|n\rangle$$

$$\left.\begin{array}{l} = (n-1)(\hat{a}|n\rangle) \\ \hat{N}|n\rangle = n|n\rangle \end{array}\right\} \Rightarrow \hat{a}|n\rangle = c(n)|n-1\rangle \quad (*)$$

$$\hat{N}(\hat{a}^+|n\rangle) = \hat{a}^+\hat{a}\,\hat{a}^+|n\rangle = \hat{a}^+(\hat{a}^+\hat{a} + \hat{I})|n\rangle$$

$$\left.\begin{array}{l} = \hat{a}^+\hat{N}|n\rangle + \hat{a}^+|n\rangle = (n+1)(\hat{a}^+|n\rangle) \\ \hat{N}|n\rangle = n|n\rangle \end{array}\right\} \Rightarrow \hat{a}^+|n\rangle = d(n)|n+1\rangle \quad (**)$$

$$(*) \Rightarrow \langle n|\hat{a}^+ = c^*(n)\langle n-1| \quad (*)'$$

$$(**) \Rightarrow \langle n|\hat{a} = d^*(n)\langle n+1| \quad (**)'$$

$$\langle n|\hat{a}^{+}a|n\rangle = |c(n)|^2 \langle n-1|n-1\rangle$$

$$\langle n|\hat{N}|n\rangle = |c(n)|^2 \Rightarrow \quad c(n) = \sqrt{n}$$

$(**) \wedge (**)' \Rightarrow$

$$\langle n|\hat{a}\hat{a}^{+}|n\rangle = |d(n)|^2$$

$$\langle n|I+\hat{a}^{+}\hat{a}|n\rangle = |d(n)|^2$$

$$1+n = |d(n)|^2 \Rightarrow \quad d(n) = \sqrt{n+1}$$

Dakle,

$$a|n\rangle = \sqrt{n}\;|n-1\rangle$$

$$a^{+}|n\rangle = \sqrt{n+1}\;|n+1\rangle$$

$a, \hat{a}^{+} \rightarrow$ Bose-ovi operatori anihilacije i kreacije

11. Napisati eksplicitno način delovanja operatora $\hat{p}_x$ i $\hat{p}_x^2$ na stanje $|\psi\rangle$ u koordinatnoj, i način delovanja operatora $\hat{x}$ i $\hat{x}^2$ na isto stanje u impulsnoj reprezentaciji.

Koordinatna reprezentacija

$$\hat{p}_x |\psi\rangle = |\chi\rangle$$

$$\hat{x}|x\rangle = x|x\rangle \quad , \quad x \in (-\infty, +\infty)$$

$$\langle x|\hat{p}_x|\psi\rangle = \langle x|\chi\rangle = \chi(x)$$

Razlaganje jedinice $\hat{I} = \int_{-\infty}^{+\infty} |x\rangle\langle x|\, dx$

$$\langle x|\hat{p}_x|\psi\rangle = \langle x|\hat{p}_x \hat{I}|\psi\rangle = \langle x|\hat{p}_x \int_{-\infty}^{+\infty} |x'\rangle\langle x'|\, dx' |\psi\rangle$$

$$= \int_{-\infty}^{+\infty} \langle x|\hat{p}_x|x'\rangle \; \psi(x')\, dx'$$

Poznato je da važi

$$\langle x|\hat{p}_x|x'\rangle = -i\hbar \; \delta(x-x') \frac{d}{dx'}$$

Dirakova delta f-ja

$$\int_{-\infty}^{+\infty} \delta(x-x')\, dx = \int_{-\infty}^{+\infty} \delta(x-x')\, dx' = 1$$

$$\int_{-\infty}^{+\infty} \delta(x-x')\, f(x)\, dx = f(x')$$

$$\langle x|\hat{p}_x|\psi\rangle = \int_{-\infty}^{+\infty} -i\hbar\; \delta(x-x') \frac{d}{dx'} \psi(x')\, dx'$$

$$\langle x|\hat{p}_x|\psi\rangle = -i\hbar \int_{-\infty} \delta(x-x') \frac{d\psi(x')}{dx'} dx'$$

$$= -i\hbar \frac{d\psi(x)}{dx}$$

Daulē, $\hat{P}_x = -i\hbar \frac{d}{dx}$

$$\hat{P}_x^2|\psi\rangle = \chi$$

$$\langle x|\hat{P}_x^2|\psi\rangle = \chi(x)$$

$$\langle x|\hat{P}_x^2|\psi\rangle = \langle x|\hat{P}_x^2 \hat{I}|\psi\rangle = \langle x|\hat{P}_x^2 \int_{-\infty}^{+\infty} |x'\rangle\langle x'| dx' |\psi\rangle$$

$$= \int_{-\infty}^{+\infty} \langle x|\hat{P}_x^2|x'\rangle \psi(x')\, dx' = \int_{-\infty}^{+\infty} \langle x|\hat{P}_x \hat{I} \hat{P}_x|x'\rangle \psi(x') dx'$$

$$= \int_{-\infty}^{+\infty} \langle x|\hat{P}_x \int_{-\infty}^{+\infty} |x''\rangle\langle x''| dx'' \hat{P}_x|x'\rangle \psi(x') dx'$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \langle x|\hat{P}_x|x''\rangle \langle x''|\hat{P}_x|x'\rangle \psi(x')\, dx' dx''$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left(-i\hbar\, \delta(x-x'') \frac{d}{dx''}\right)\left(-i\hbar\, \delta(x''-x') \frac{d}{dx'}\right) \psi(x') dx' dx''$$

$$= -\hbar^2 \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \delta(x-x'')\, \delta(x''-x') \frac{d}{dx''} \frac{d}{dx'} \psi(x')\, dx' dx''$$

$x'' = x'$ unidaușā integrala

$$-\hbar^2 \int_{-\infty}^{+\infty} \delta(x-x') \frac{d^2\psi(x')}{dx'^2} dx'$$

$$= -\hbar^2 \frac{d^2\psi(x)}{dx^2} \Rightarrow \hat{P}_x^2 = -\hbar^2 \frac{d^2}{dx^2}$$

Impulsna reprezentacija

$$\hat{x}\,|\psi\rangle = |\varphi\rangle \qquad \hat{P}_x\,|P_x\rangle = P_x\,|P_x\rangle,\ P_x \in (-\infty, +\infty)$$

$$\langle P_x|\hat{x}|\psi\rangle = \langle P_x|\varphi\rangle = \varphi(P_x)$$

$$\langle P_x|\hat{x}\,|\int_{-\infty}^{+\infty}|P_x'\rangle\langle P_x'|\,dp_x'\,|\psi\rangle =$$

$$\int_{-\infty}^{+\infty}\langle P_x|\hat{x}|P_x'\rangle\,\psi(P_x')\,dP_x' = i\hbar\int_{-\infty}^{+\infty}\delta(P_x-P_x')\,\frac{d}{dP_x'}\psi(P_x')\,dP_x'$$

$$= i\hbar\,\frac{d\psi(P_x)}{dP_x}$$

Dakle $\quad i\hbar\,\dfrac{d\psi(P_x)}{dP_x} = \varphi(P_x) \Rightarrow \hat{x} = i\hbar\,\dfrac{d}{dP_x}$

Delovanje $\hat{x}^2$ za domaći, po analogiji sa prethodnim za $\hat{P}_x^2$ u koordinatnoj reprezentaciji.

Primedba

Ovde u zadacima se ne vidi sled

$$[\hat{x}, \hat{P}_x] = i\hbar \qquad\qquad [\hat{x}, \hat{P}_x] = i\hbar$$

$$\downarrow \qquad\qquad \hat{x} \to i\hbar\frac{d}{dP_x}$$

$$\hat{P}_x \to -i\hbar\frac{d}{dx} \qquad\qquad \downarrow$$

$$\downarrow$$

$$\langle x|\hat{P}_x|x'\rangle = -i\hbar\,\delta(x-x')\frac{d}{dx} \qquad\qquad \langle P_x|\hat{x}|P_x'\rangle = i\hbar\,\delta(P_x-P_x')\frac{d}{dP_x}$$

Razrešiti!

10. Rešiti svojstveni problem opservable impulsa $\hat{p}_x$ u koordinatnoj reprezentaciji. Uopštiti na trodimenzionalni slučaj.

$$\hat{p}_x |p_x\rangle = p_x |p_x\rangle \qquad p_x \in (-\infty, +\infty)$$

U koordinatnoj reprezentaciji

$$\hat{p}_x \Rightarrow -i\hbar \frac{d}{dx}$$

$$|p_x\rangle \to \psi_{p_x}(x) \qquad (|\psi_{p_x}\rangle \equiv |p_x\rangle)$$

$$-i\hbar \frac{d\psi_{p_x}(x)}{dx} = p_x \psi_{p_x}(x)$$

$$\frac{d\psi_{p_x}(x)}{\psi_{p_x}(x)} = \frac{i}{\hbar} p_x dx \quad / \int$$

$$\int \frac{d\psi_{p_x}(x)}{\psi_{p_x}(x)} = \frac{i}{\hbar} \int p_x dx$$

$$\ln \psi_{p_x}(x) = \frac{i}{\hbar} p_x x + \ln C \longrightarrow \boxed{\text{kompleksan broj}}$$

$$\boxed{\psi_{p_x}(x) = C e^{\frac{i}{\hbar} x p_x}}$$

$|\psi_{p_x}\rangle \in U(\mathcal{H})$ uopšteni Hilbertov prostor

$$\langle \psi_{p_x} | \psi_{p_x'} \rangle = \delta(p_x - p_x')$$

$$\langle \psi_{p_x} | \psi_{p_x'} \rangle = \langle \psi_{p_x} | \hat{I} | \psi_{p_x'} \rangle = \langle \psi_{p_x} | \int_{-\infty}^{+\infty} |x\rangle\langle x| dx | \psi_{p_x'} \rangle$$

$$\langle \psi_{p_x} | \psi_{p_x'} \rangle = \int_{-\infty}^{+\infty} \langle \psi_{p_x} | x \rangle \langle x | \psi_{p_x'} | \, dx$$

$$= \int_{-\infty}^{+\infty} \psi^*_{p_x}(x) \psi_{p_x'}(x) dx = |C|^2 \int_{-\infty}^{+\infty} e^{-\frac{i}{\hbar} x p_x} e^{\frac{i}{\hbar} x p_x'} dx$$

$$= |C|^2 \int_{-\infty}^{+\infty} e^{-\frac{i}{\hbar} x (p_x - p_x')} dx$$

Reprezentacija delta f-je

$$\delta(p_x - p_x') = \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} e^{-\frac{i}{\hbar} x (p_x - p_x')} dx$$

$$C = \frac{1}{\sqrt{2\pi\hbar}}$$

$$\psi_{p_x}(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar} x p_x}$$

Za trodimenzionalni slučaj

$$\hat{\vec{p}} | \vec{p} \rangle = \vec{p} | \vec{p} \rangle$$

$$| \vec{p} \rangle = | p_x \rangle \otimes | p_y \rangle \otimes | p_z \rangle \equiv | \psi_{p_x} \rangle \otimes | \psi_{p_y} \rangle \otimes | \psi_{p_z} \rangle$$

$$\psi_{\vec{p}}(\vec{r}) = \psi_{p_x}(x) \psi_{p_y}(y) \psi_{p_z}(z)$$

$$= \frac{1}{(2\pi\hbar)^{3/2}} e^{\frac{i}{\hbar} \vec{r} \cdot \vec{p}}$$

Za domaći:

Rešiti svojstveni problem opservable $\hat{x}$ u impulsnoj reprezentaciji. Uopštiti na 3D.

$\hat{x}|x\rangle = x|x\rangle$

$\hat{x} \to i\hbar \frac{d}{dp_x}$ i koristiti da je

$|x\rangle \to \psi_x(p_x)$ $\delta(x-x') = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} e^{\frac{i}{\hbar}(x-x')p_x}\, dp_x$

$$\psi_x(p_x) = \frac{1}{\sqrt{2\pi\hbar}}\, e^{-\frac{i}{\hbar}xp_x}$$

1. Rešiti svojstveni problem Hamiltonijana za slobodnu trodimenzionalnu česticu mase m u koordinatnoj reprezentaciji

$$\hat{H} = \frac{\hat{\vec{p}}^{\,2}}{2m} \quad , \quad \hat{H}|\psi\rangle = E|\psi\rangle$$

Prvo razmatramo 1 D slučaj.

$$\hat{H} = \frac{\hat{p}_x^2}{2m} \qquad \hat{H}_x|\varphi\rangle = E_x|\varphi\rangle$$

Sv. problem

$$\hat{p}_x^2 \rightarrow -\hbar^2 \frac{d^2}{dx^2}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\varphi(x)}{dx^2} = E_x\varphi(x)$$

$$\varphi''(x) = -\frac{2mE_x}{\hbar^2}\varphi(x) \quad ; \quad \omega_x^2 = \frac{2mE_x}{\hbar^2}$$

$$\varphi''(x) + \omega_x^2\varphi(x) = 0$$

$$\varphi(x) = c_1 e^{-i\omega_x x} + c_2 e^{i\omega_x x} \qquad (*)$$

Ali $[\hat{H}_x, \hat{p}_x] = 0 \Rightarrow$ zajednički svojstveni vektori

(*) nije svojstvena f-ja $\hat{p}_x$

$\hat{p}_x\varphi(x) \neq \lambda\varphi(x)$ ali jeste za $\hat{p}_x^2$

Zajedničke svojstvene f-je za $\hat{p}_x$ i $\hat{p}_x^2$ su $e^{\pm i\omega_x x}$. U prethodnom zadatku je nadjeno svojstveno rešenje za $\hat{p}_x \rightarrow \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar} x p_x}$

Dakle, iz prethodnog zadatka

$$\varphi(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar} x p_x}$$

Ovde $\varphi(x) \equiv C \, e^{i\omega_x x}$

Poređenjem, vidimo da je $\omega_x = \frac{p_x}{\hbar} \Rightarrow$

$$\omega_x^2 = \frac{p_x^2}{\hbar^2} \Rightarrow \frac{2mE_x}{\hbar^2} = \frac{p_x^2}{\hbar^2} \Rightarrow p_x^2 = 2mE_x \Rightarrow$$

$$E_x = \frac{p_x^2}{2m}$$

Dakle svojstvena f-ja Ψ za 1D slobodnu česticu je $\varphi(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{i\omega_x x}$ a svojstvena vrednost

$$E_x = \frac{p_x^2}{2m}$$

Uopštenje na 3D : $\mathcal{H}^{(u)} = \mathcal{H}^{(x)} \otimes \mathcal{H}^{(y)} \otimes \mathcal{H}^{(z)}$

$$\hat{H} = \frac{\vec{p}^2}{2m} = \frac{1}{2m} \left(\hat{p}_x^2 \otimes \hat{I}_y \otimes \hat{I}_z + \hat{I}_x \otimes \hat{p}_y^2 \otimes \hat{I}_z + \hat{I}_x \otimes \hat{I}_y \otimes \hat{p}_z^2 \right)$$

$$= \hat{H}_x + \hat{H}_y + \hat{H}_z$$

$$\hat{H}_x |\varphi\rangle = E_x |\varphi\rangle$$

$$\hat{H}_y |\chi\rangle = E_y |\chi\rangle \qquad |\psi\rangle = |\varphi\rangle |\chi\rangle |\lambda\rangle$$

$$\hat{H}_z |\lambda\rangle = E_z |\lambda\rangle$$

i je $\psi(\vec{r}) = \frac{1}{\cancel{(2\pi\hbar)}^{\frac{3}{2}}} e^{i\vec{\omega}\cdot\vec{r}}$ i $E = \frac{\vec{p}^2}{2m}$

$$= \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}$$

Eksplicitnim računom u koordinatnoj reprezentaciji potvrditi komutacione relacije

a) $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$, $[\hat{x}_i, \hat{x}_j] = 0 = [\hat{p}_i, \hat{p}_j]$

b) $[\hat{L}_i, \hat{L}_j] = i\hbar\epsilon_{ijk}\hat{L}_k$

---

a) $\hat{x}_i \to x_i \qquad \hat{p}_j \to -i\hbar\frac{\partial}{\partial x_j}$

$[\hat{x}_i, \hat{x}_j] \to [x_i, x_j] = 0$

$$[\hat{p}_i, \hat{p}_j]|\psi\rangle \to \left[-i\hbar\frac{\partial}{\partial x_i}, -i\hbar\frac{\partial}{\partial x_j}\right]\psi(\vec{r}) =$$

$$= (i\hbar)^2\left(\frac{\partial}{\partial x_i}\frac{\partial\psi(\vec{r})}{\partial x_j} - \frac{\partial}{\partial x_j}\frac{\partial\psi(\vec{r})}{\partial x_i}\right) =$$

$$= (i\hbar)^2\left(\frac{\partial^2\psi(\vec{r})}{\partial x_i\partial x_j} - \frac{\partial^2\psi(\vec{r})}{\partial x_j\partial x_i}\right) = 0$$

$$[\hat{x}_i, \hat{p}_j]|\psi\rangle \longrightarrow \left[x_i, -i\hbar\frac{\partial}{\partial x_j}\right]\psi(\vec{r}) =$$

$$= \underbrace{\cancel{(i\hbar)^2}}_{-i\hbar}\left(x_i\frac{\partial\psi(\vec{r})}{\partial x_j} - \frac{\partial}{\partial x_j}(x_i\psi(\vec{r}))\right) =$$

$$= \underbrace{\cancel{(i\hbar)^2}}_{-i\hbar}\left(x_i\frac{\partial\psi(\vec{r})}{\partial x_j} - \frac{\partial x_i}{\partial x_j}\psi(\vec{r}) - x_i\frac{\partial\psi(\vec{r})}{\partial x_j}\right)$$

$$= i\hbar\delta_{ij}\psi(\vec{r}) \to i\hbar\delta_{ij}|\psi\rangle$$

$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$

D) $L_i = \epsilon_{ipq} \hat{X}_p P_q$

$\hat{L}_j = \epsilon_{jmn} \hat{X}_m \hat{p}_n$

$$[\hat{L}_i, \hat{L}_j]\,|\psi\rangle \to \left[\epsilon_{ipq} x_p \left(-i\hbar \frac{\partial}{\partial x_q}\right), \epsilon_{jmn} x_m \left(-i\hbar \frac{\partial}{\partial x_n}\right)\right]\psi(\vec{r})$$

$$= (i\hbar)^2 \epsilon_{ipq}\epsilon_{jmn} \left[x_p \frac{\partial}{\partial x_q}, x_m \frac{\partial}{\partial x_n} \right] \psi(\vec{r})$$

$$= (i\hbar)^2 \epsilon_{ipq}\epsilon_{jmn} \left[x_p \frac{\partial}{\partial x_q}\left(x_m \frac{\partial \psi(\vec{r})}{\partial x_n}\right) - x_m \frac{\partial}{\partial x_n}\left(x_p \frac{\partial \psi(\vec{r})}{\partial x_q}\right) \right]$$

$$= (i\hbar)^2 \epsilon_{ipq}\epsilon_{jmn} \left(x_p \left\{ \frac{\partial x_m}{\partial x_q} \frac{\partial \psi(\vec{r})}{\partial x_n} + x_m \frac{\partial^2 \psi(\vec{r})}{\partial x_q \partial x_n} \right\} - \right.$$

$$\left. - x_m \left\{ \frac{\partial x_p}{\partial x_n} \frac{\partial \psi(\vec{r})}{\partial x_q} + x_p \frac{\partial^2 \psi(\vec{r})}{\partial x_n \partial x_q} \right\} \right) =$$

$$= (i\hbar)^2 \epsilon_{ipq}\epsilon_{jmn} \left(x_p \delta_{mq} \frac{\partial \psi(\vec{r})}{\partial x_n} + \cancel{x_p x_m \frac{\partial^2 \psi(\vec{r})}{\partial x_q \partial x_n}} - \right.$$

$$\left. x_m \delta_{pn} \frac{\partial \psi(\vec{r})}{\partial x_q} - \cancel{x_m x_p \frac{\partial^2 \psi(\vec{r})}{\partial x_n \partial x_q}} \right)$$

$$= (i\hbar)^2 \epsilon_{ipq}\epsilon_{jmn} \left(x_p \delta_{mq} \frac{\partial \psi(\vec{r})}{\partial x_n} - x_m \delta_{pn} \frac{\partial \psi(\vec{r})}{\partial x_q} \right)$$

$$(i\hbar)^2 \epsilon_{ipq}\epsilon_{jmn}\, x_p \delta_{mq} \frac{\partial \psi(\vec{r})}{\partial x_n} -$$

$$-(i\hbar)^2 \epsilon_{ipq}\epsilon_{jmn}\, x_m \delta_{pn} \frac{\partial \psi(\vec{r})}{\partial x_q}$$

$$(i\hbar)^2 \epsilon_{ipq}\epsilon_{jqn}\, x_p \frac{\partial \psi(\vec{r})}{\partial x_n} -$$

$$-(i\hbar)^2 \epsilon_{inq}\epsilon_{jmn}\, x_m \frac{\partial \psi(\vec{r})}{\partial x_q}$$

$$= -(i\hbar)\, \underline{\epsilon_{qip}\, \epsilon_{qjn}}\, x_p \frac{\partial \psi(\vec{r})}{\partial x_n} +$$
$$+ (i\hbar)^2\, \underline{\epsilon_{niq}\, \epsilon_{njm}}\, x_m \frac{\partial \psi(\vec{r})}{\partial x_q}$$

$$\downarrow \begin{matrix} i & p \\ j & n \end{matrix} \downarrow - \begin{matrix} i & p \\ j & n \end{matrix}, \quad \downarrow \begin{matrix} i & q \\ j & m \end{matrix} \downarrow - \begin{matrix} i & q \\ j & m \end{matrix} \leftarrow$$

mnemoničko pravilo za $\epsilon_{ijk}\, \epsilon_{ipq}$

$$= -(i\hbar)^2 (\delta_{ij}\delta_{pn} - \delta_{in}\delta_{pj})\, x_p \frac{\partial \psi(\vec{r})}{\partial x_n}$$
$$+ (i\hbar)^2 (\delta_{ij}\delta_{qm} - \delta_{im}\delta_{qj})\, x_m \frac{\partial \psi(\vec{r})}{\partial x_q}$$

$$= \cancel{-(i\hbar)^2 \delta_{ij} x_n \frac{\partial \psi}{\partial x_n}} + (i\hbar)^2 x_j \frac{\partial \psi}{\partial x_i} + \cancel{(i\hbar)^2 \delta_{ij} x_m \frac{\partial \psi}{\partial x_m}} -$$
$$- (i\hbar)^2 x_i \frac{\partial \psi}{\partial x_j}$$

$$= (i\hbar)^2 \left(x_j \frac{\partial \psi}{\partial x_i} - x_i \frac{\partial \psi}{\partial x_j} \right)$$

$$= i\hbar \left(x_j \left(i\hbar \frac{\partial \psi}{\partial x_i} \right) - x_i \left(i\hbar \frac{\partial \psi}{\partial x_j} \right) \right)$$

$$= i\hbar \left(x_i \left(-i\hbar \frac{\partial \psi}{\partial x_j} \right) - x_j \left(-i\hbar \frac{\partial \psi}{\partial x_i} \right) \right)$$

$$= i\hbar \left(x_i \left(-i\hbar \frac{\partial}{\partial x_j} \right) - x_j \left(-i\hbar \frac{\partial}{\partial x_i} \right) \right) \psi(\vec{r}) \longrightarrow$$

$$\downarrow \begin{matrix} i & j \\ m & n \end{matrix} \downarrow$$

$$\rightarrow i\hbar \left(\hat{x}_i \hat{p}_j - \hat{x}_j \hat{p}_i \right) |\psi\rangle =$$

$$= i\hbar \sum_{m,n} (\delta_{im}\delta_{jn} - \delta_{jm}\delta_{in})\, \hat{x}_m \hat{p}_n\, |\psi\rangle =$$

$$= i\hbar \sum_s \epsilon_{sij}\, \underbrace{\epsilon_{smn}\, \hat{x}_m \hat{p}_n}\, |\psi\rangle$$

$$= i\hbar \sum_s \epsilon_{sij}\, \hat{L}_s\, |\psi\rangle \Rightarrow \underline{[\hat{L}_i, \hat{L}_j] = i\hbar\, \epsilon_{sij}\, \hat{L}_s}$$

Dato stanje $|\psi\rangle$ zadato je u diskretnoj reprezentaciji. Naći reprezentaciju tog stanja u:

a) drugoj diskretnoj reprezentaciji $|\chi_i\rangle$

b) ~~drugoj~~ datoj kontinualnoj reprezentaciji $|a\rangle$

$|\varphi_m\rangle$ – bazis koji def. stanje $|\psi\rangle$

Reprezentacija znači izbor bazisa. Ako je bazis u HP prostoru to je diskretna reprezentacija a a ako je u uopštenom HP onda je kontinualna reprezentacija.

a) $|\psi\rangle = \sum_m c_m |\varphi_m\rangle$

$$c_m = \langle \varphi_m | \psi \rangle$$

U drugom bazisu

$$|\psi\rangle = \sum_i b_i |\chi_i\rangle \qquad b_i = \langle \chi_i | \psi \rangle$$

veza između c_m i b_i

$$b_i = \langle \chi_i | \psi \rangle = \langle \chi_i | \hat{I} | \psi \rangle = \langle \chi_i | \sum_m |\varphi_m\rangle \langle \varphi_m | \psi \rangle$$

$$= \sum_m \langle \chi_i | \varphi_m \rangle c_m = \sum_m d_{im} c_m$$

$\boxed{d_{im}}$ su elementi matrice prelaza između dva bazisa!

b) kontinualni bazis

$$|a\rangle \in U(\mathcal{H})$$

$$|\psi\rangle = \hat{I}|\psi\rangle = \int |a\rangle da \langle a | \psi \rangle = \int \psi(a) |a\rangle da$$

Veza sa starim zapisom

$$\psi(a) = \langle a | I | \psi\rangle = \langle a | \sum_m |\varphi\rangle_m \langle\varphi|\psi\rangle$$

$$= \sum_m \langle a|\varphi_m\rangle c_m$$

Елегантније

Задато

$$|\psi\rangle = \sum_m c_m |\psi_m\rangle$$

У другој дискр. репр. $|\chi_i\rangle$

$$\hat{I}_\chi = \sum_i |\chi_i\rangle\langle\chi_i|$$

$$|\psi\rangle = \sum_m c_m \hat{I}_\chi |\psi_m\rangle = \sum_m c_m \sum_i |\chi_i\rangle\langle\chi_i|\psi_m\rangle$$

$$= \sum_{i,m} c_m \langle\chi_i|\psi_m\rangle |\chi_i\rangle = \sum_{i,m} c_m \underset{\uparrow}{\alpha_{im}} |\chi_i\rangle$$

матрица прелаза са базис на базис

У датој континуалној репр. $|a\rangle \Rightarrow \hat{I}_a = \int |a\rangle\langle a| da$

$$|\psi\rangle = \sum_m c_m \hat{I}_a |\psi_m\rangle = \sum_m \int c_m \langle a|\psi_m\rangle |a\rangle da$$

$$= \sum_m \int c_m \psi_m(a) |a\rangle da$$

Zadato je stanje $|\psi\rangle$ trodimenzionalne čestice u $\{\hat{x}, \hat{p}_y, \hat{z}\}$ reprezentaciji. Naći isto stanje u $\{\hat{p}_x, \hat{y}, \hat{z}\}$ reprezentaciji, korišćenjem odgovarajućih Furijeovih transforma.

$|\psi\rangle$

$\psi(x, p_y, z) \leftarrow$ zadato : $\mathcal{H}^{(x)} \otimes \mathcal{H}^{(y)} \otimes \mathcal{H}^{(z)}$

$$\psi(p_x, y, z) = \left(\langle p_x| \otimes \langle y| \otimes \langle z|\right)|\psi\rangle =$$

$$= \langle p_x| \otimes \langle y| \otimes \langle z| \hat{I}_x \otimes \hat{I}_y \otimes \hat{I}_z |\psi\rangle$$

$$= \langle p_x|\langle y|\langle z| \int_{-\infty}^{+\infty} |x\rangle\langle x| dx \int_{-\infty}^{+\infty} |p_y\rangle\langle p_y| dp_y \int_{-\infty}^{+\infty} |z'\rangle\langle z'| dz' |\psi\rangle$$

$$= \iint_{-\infty}^{+\infty} \langle p_x|x\rangle \langle y|p_y\rangle \underbrace{\langle x| \otimes \langle p_y| \quad \langle z|\psi\rangle}\, dx\, dp_y$$

$$\langle y|p_y\rangle = \psi_{p_y}(y) = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar} y p_y}$$

$$\langle x|p_x\rangle = \psi_{p_x}(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar} x p_x}$$

$$\psi(p_x, y, z) = \frac{1}{2\pi\hbar} \iint_{-\infty}^{+\infty} e^{\frac{i}{\hbar}(y p_y - x p_x)} \underline{\psi(x, p_y, z)}\, dx\, dp_y$$

zadato

Operator A je zadat u diskretnoj reprezentaciji $\{A_{ij}\}$. Naći predstavljanje ovog operatora u:

a) drugoj diskretnoj reprezentaciji $|\chi_m\rangle \in \mathcal{H}$

b) zadatoj kontinualnoj reprezentaciji $|a\rangle \in \mathcal{H}$

$\hat{A}$ je zadato u diskretnoj reprezentaciji $|\varphi_i\rangle$

$$A_{ij} = \langle \varphi_i | \hat{A} | \varphi_j \rangle$$

a) $$\langle \chi_m | \hat{A} | \chi_n \rangle = \langle \chi_m | \hat{I} \hat{A} \hat{I} | \chi_n \rangle$$

$$= \langle \chi_m | \sum_i |\varphi_i\rangle\langle\varphi_i| \hat{A} | \sum_j |\varphi_j\rangle\langle\varphi_j| \chi_n \rangle$$

$$= \sum_{i,j} \langle \chi_m | \varphi_i \rangle \langle \varphi_i | \hat{A} | \varphi_j \rangle \langle \varphi_j | \chi_n \rangle$$

$$= \sum_{i,j} \alpha_{mi} A_{ij} \alpha_{jn}^*$$

$$\underset{\text{novo}}{A_{mn}} = \sum_{i,j} \alpha_{mi} \underset{\text{staro}}{A_{ij}} \alpha_{jn}^*$$

b) $|a\rangle$

$$A(a,a') = \langle a | \hat{A} | a' \rangle = \langle a | \sum_i |\varphi_i\rangle\langle\varphi_i| A | \sum_j |\varphi_j\rangle\langle\varphi_j|a'\rangle$$

$$A(a,a') = \sum_{i,j} \varphi_i(a) \, A_{ij} \, \varphi_j^*(a)$$

novo $\qquad$ staro

$$|\psi\rangle = \sum_i c_i \, |\varphi_i\rangle \, ; \; c_i = \langle \varphi_i | \psi \rangle$$

$$|\psi\rangle = \int \psi(a) \, |a\rangle \, da$$

Razvoj stanja po bazisima

$$\psi(a) = \langle a | \psi \rangle$$

Talasna f-ja

Конзистентније

дискр. $|x_i\rangle \to \hat{I}_x$

задато $\hat{A} = \sum_{i,j} A_{ij} |\varphi_i\rangle\langle\varphi_j|$

$$\hat{A} = \sum_{i,j} A_{ij} \, \hat{I}_x |\varphi_i\rangle\langle\varphi_j| \hat{I}_x$$

$$\hat{A} = \sum_{i,j} A_{ij} \sum_m |x_m\rangle\langle x_m|\varphi_i\rangle\langle\varphi_j| \sum_n |x_n\rangle\langle x_n|$$

$$\hat{A} = \sum_{i,j} \sum_{m,n} A_{ij} \, d_{mi} \, d_{jn}^* \, |x_m\rangle\langle x_n|$$

конт $|a\rangle \to \hat{I}_a$

$$\hat{A} = \sum_{i,j} A_{ij} \, |\varphi_i\rangle\langle\varphi_j|$$

$$\hat{A} = \sum_{i,j} A_{ij} \, \hat{I}_a |\varphi_i\rangle\langle\varphi_j| \hat{I}_a$$

$$\hat{A} = \sum_{i,j} \iint A_{ij} \, \varphi_i(a) \varphi_j^*(a') \, |a\rangle\langle a'| \, da \, da'$$

крупна

$$A(a,a') = \sum_{i,j} \varphi_i(a) A_{ij} \varphi_j^*(a')$$

Operator A je zadat u kontinualnoj reprezentaciji $A(x,x')$. Naći predstavljanje ovog operatora u:

a) nekoj diskretnoj reprezentaciji $|\chi_m\rangle \in \mathcal{H}$

b) drugoj kontinualnoj reprezentaciji $|y\rangle \in U(\mathcal{H})$

$$A(x,x') = \langle x|\hat{A}|x'\rangle \quad , |x\rangle \in U(\mathcal{H})$$

a) $$\langle \chi_m|\hat{A}|\chi_n\rangle = \langle \chi_m|\hat{I}\hat{A}\hat{I}|\chi_n\rangle$$

$$= \langle \chi_m| \int |x\rangle\langle x| dx \, \hat{A} \int |x'\rangle\langle x'| dx' \, |\chi_n\rangle$$

$$= \iint \chi_m^*(x) \, A(x,x') \, \chi_n(x') \, dx dx'$$

$$A_{nm} = \iint \chi_m^*(x) \, A(x,x') \, \chi_m(x') \, dx dx'$$

b) $$\langle y|A|y'\rangle = \langle y|\hat{I}\hat{A}\hat{I}|y'\rangle =$$

$$= \langle y| \int |x\rangle\langle x| dx \, \hat{A} \int |x'\rangle\langle x'| dx' \, |y'\rangle$$

$$= \iint y^*(x) \, A(x,x') \, y'(x') \, dx dx'$$

Dokazati da je "trag" operatora $\hat{A}$,

$$\mathrm{tr}\hat{A} = \sum_i \langle i|\hat{A}|i\rangle \quad (\langle i|j\rangle = \delta_{ij})$$

a) reprezentaciono invarijantna veličina

b) $\mathrm{tr}(a\hat{A}) = a\,\mathrm{tr}(\hat{A})$

c) $\mathrm{tr}(\hat{A}\hat{B}) = \mathrm{tr}(\hat{B}\hat{A})$

d) $\mathrm{tr}(\hat{A}+\hat{B}) = \mathrm{tr}\hat{A} + \mathrm{tr}\hat{B}$

---

a) $$\mathrm{tr}\hat{A} = \sum_i \langle i|\hat{A}|i\rangle = \sum_i \langle i|\hat{I}\hat{A}\hat{I}|i\rangle$$

$$= \sum_i \langle i| \sum_l |l\rangle\langle l| \hat{A} \sum_m |m\rangle\langle m|i\rangle$$

$$= \sum_i \sum_l \sum_m \langle i|l\rangle \langle l|\hat{A}|m\rangle\langle m|i\rangle$$

$$= \sum_{i,l,m} \langle l|\hat{A}|m\rangle \langle m|i\rangle\langle i|l\rangle$$

$$= \sum_{l,m} \langle l|\hat{A}|m\rangle \langle m| \underbrace{\sum_i |i\rangle\langle i|}_{I} l\rangle$$

$$= \sum_{l,m} \langle l|\hat{A}|m\rangle \delta_{lm} = \sum_m \langle m|\hat{A}|m\rangle$$

b) $$\mathrm{tr}(a\hat{A}) = \sum_i \langle i|a\hat{A}|i\rangle = a\sum_i \langle i|\hat{A}|i\rangle = a\,\mathrm{tr}\hat{A}$$

c) $$\mathrm{tr}\hat{A}\hat{B} = \sum_i \langle i|\hat{A}\hat{B}|i\rangle = \sum_i \langle i|\hat{A}\hat{I}\hat{B}|i\rangle$$

$$\sum_i \langle i | A \sum_j |j\rangle\langle j| \hat{B} | i \rangle = \sum_{i,j} \langle i|\hat{A}|j\rangle \langle j|B|i\rangle$$

$$= \sum_{i,j} \langle j|B|i\rangle \langle i|A|j\rangle = \sum_j \langle j|\hat{B} \sum_i |i\rangle\langle i|A|j\rangle$$

$$= \sum_j \langle j|\hat{B}\hat{A}|j\rangle = \operatorname{tr} \hat{B}\hat{A}$$

d) $\operatorname{tr}(\hat{A}+\hat{B}) = \sum_i \langle i|\hat{A}+\hat{B}|i\rangle =$

$$= \sum_i \langle i|\hat{A}|i\rangle + \sum_i \langle i|\hat{B}|i\rangle = \operatorname{tr}\hat{A} + \operatorname{tr}\hat{B}$$

26. Naći uslove ermitičnosti opservable $\hat{P}_x$ u koordinatnoj reprezentaciji.

Ermitski operator $\hat{M}^+ = \hat{M}$

$\langle g|\hat{M}|f\rangle \rightarrow$ skalar

$$(\langle g|\hat{M}|f\rangle)^+ \equiv (\langle g|\hat{M}|f\rangle)^*$$

Sa druge strane

$$(\langle g|\hat{M}|f\rangle)^+ = \langle f|\hat{M}^+|g\rangle = \langle f|\hat{M}|g\rangle$$

što sve zajedno daje

$$\langle f|\hat{M}|g\rangle = \langle g|M|f\rangle^*$$

Kada uslov za ermitičnost opservable

$\hat{M} = \hat{P}_x$

$$\langle f|\hat{P}_x|g\rangle = \langle g|\hat{P}_x|f\rangle^*$$

odnosno

$$\iint_{-\infty}^{+\infty} f^*(x)\langle x|\hat{P}_x|x'\rangle g(x')\,dx\,dx' = \left(\iint_{-\infty}^{+\infty} g^*(x)\langle x|\hat{P}_x|x'\rangle f^*(x')\,dx\,dx'\right)^*$$

$$\iint_{-\infty}^{+\infty} f^*(x)\left(-i\hbar\,\delta(x-x')\frac{d}{dx'}\right)g(x')\,dx\,dx' = \left(\iint_{-\infty}^{+\infty} g^*(x)\left(-i\hbar\,\delta(x-x')\right)\frac{d f(x')}{dx'}\,dx\,dx'\right)^*$$

$$-i\hbar\int_{-\infty}^{+\infty} f^*(x)\frac{dg(x)}{dx}\,dx = i\hbar\int_{-\infty}^{+\infty} g(x)\frac{df^*(x)}{dx}\,dx$$

$$\left\lceil \; d(f^* g) = f^* dg + g\, df^* \right.$$

$$\left. g\, df^* = d(f^* g) - f^* dg \;\right\rfloor$$

$$-i\hbar \int_{-\infty}^{+\infty} f^*(x)\, dg(x) = i\hbar \int_{-\infty}^{+\infty} g(x)\, df^*(x)$$

$$-i\hbar \int_{-\infty}^{+\infty} f^*(x)\, dg(x) = i\hbar \int_{-\infty}^{+\infty} [d(f^* g) - f^*(x)\, dg(x)]$$

$$= -i\hbar \int_{-\infty}^{+\infty} f^*(x)\, dg(x) + i\hbar \int_{-\infty}^{+\infty} d(f^* g)$$

L.S. = D.S ako je

$$\int_{-\infty}^{+\infty} d(f^*(x) g(x)) = 0$$

$$f^*(x) g(x) \Big|_{-\infty}^{+\infty} = 0$$

$$\lim_{x \to \pm\infty} g(x) = 0$$

$\hat{P}_x$ je ermitski operator na domenu ovakvih f-ja

U sferno-polarnim koordinatama zadaba je operatora $\hat{p}_r$ koja sa radijalnom komponentom opservable položaja, $\hat{r} \equiv |\hat{\vec{r}}|$, zadovoljava komutacionu relaciju $[\hat{r}, \hat{p}_r] = i\hbar$ i koja je u koordinatnoj reprezentaciji data operatorom $p_r = -i\hbar\left(\frac{1}{r} + \frac{\partial}{\partial r}\right)$. Naći uslove njene ermitičnosti u Hilbertovom prostoru stanja $\mathcal{H}^{(r)}$ u kojem je skalarni proizvod dvaju f-ja $f(r)$ i $g(r)$ u koordinatnoj reprezentaciji, zadat integralom $\int_0^{+\infty} f^*(r)\, g(r)\, r^2 dr$.

$$\langle f | \hat{M} | g \rangle = (\langle g | \hat{M} | f \rangle)^*$$

?

$$\hat{M} = \hat{p}_r$$

$$\langle f | \hat{p}_r | g \rangle = \langle f | \hat{I} \hat{p}_r \hat{I} | g \rangle =$$

$$\hat{I} = \int_0^{+\infty} |r\rangle\langle r| \, r^2 dr$$

Jer

$$\hat{I} = \hat{I}_x \otimes \hat{I}_y \otimes \hat{I}_z = \iiint_{-\infty}^{+\infty} |x\rangle|y\rangle|z\rangle \otimes \langle x|\langle y|\langle z| \, dx dy dz$$

$$|\vec{r}\rangle = |x\rangle|y\rangle|z\rangle = |r\rangle|\theta\rangle|\varphi\rangle$$

$$dx dy dz = r^2 \sin\theta \, d\varphi$$

$$= \int_0^{+\infty}\int_0^{2\pi}\int_0^{\pi} |r\rangle|\theta\rangle|\varphi\rangle \otimes \langle\varphi|\langle\theta|\langle r| \; r^2 \sin\theta \, dr d\theta d\varphi$$

$$= \hat{I}_r \otimes \hat{I}_\theta \otimes \hat{I}_\varphi$$

$$\langle \vec{r} | \vec{p} | \vec{r}\,' \rangle = -i\hbar \, \delta(\vec{r}-\vec{r}\,') \frac{\partial}{\partial \vec{r}}$$

$$\delta(\vec{r}-\vec{r}\,') = \frac{1}{r^2 \sin\theta} \delta(r-r') \, \delta(\theta-\theta') \, \delta(\varphi-\varphi')$$

$$\frac{\partial}{\partial \vec{r}} \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \longrightarrow \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \varphi} \right)$$

pa je

$$\boxed{\langle r | \hat{p}_r | r' \rangle = -i\hbar \, \frac{\delta(r-r')}{r^2} \left(\frac{1}{r} + \frac{\partial}{\partial r} \right)}$$

nastavak

$$= \iint_0^{+\infty} \langle f | r \rangle \langle r | r^2 dr \, \hat{p}_r | r' \rangle \langle r' | r'^2 dr' | g \rangle$$

$$\iint_0^{+\infty} f^*(r) \langle r | \hat{p}_r | r' \rangle g(r') \; r^2 r'^2 \, dr \, dr'$$

$$= \iint_0^{+\infty} f^*(r) \left[\frac{i\hbar}{r^2} \delta(r-r') \left(\frac{1}{r} + \frac{\partial}{\partial r} \right) g(r') \right] r^2 r'^2 \, dr \, dr'$$

$$r' \to r$$

$$= (-i\hbar) \int_0^{+\infty} f^*(r) \frac{1}{r^2} \left(\frac{1}{r} + \frac{\partial}{\partial r} \right) g(r) \, r^4 \, dr$$

$$(-i\hbar) \int_0^{+\infty} f^*(r) \left(\frac{1}{r} + \frac{1}{\partial r} \right) g(r) \; r^2 \, dr$$

A КРВИЧТИ ОДЗВОЈ ?

$$\langle g | \hat{p}_r | f \rangle)^* = \left[(-i\hbar) \int_0^{+\infty} g^*(r) \left(\frac{1}{r} + \frac{\partial}{\partial r} \right) f(r) \; r^2 dr \right]^*$$

$$= +i\hbar \int_0^{\infty} \cancel{g^*}(r) \left(\frac{1}{r} + \frac{\partial}{\partial r} \right) f^*(r) \, r^2 \, dr$$

$$= i\hbar \int_0 g(r) f^*(r)\, r\, dr + i\hbar \int_0^{+\infty} g(r) \frac{\partial f^*(r)}{\partial r} r^2 dr$$

$$U = r g(r) f^*(r) \qquad dU = \left[g(r) f^*(r) + r \frac{\partial g(r)}{\partial r} f^*(r) + r g(r) \frac{\partial f^*(r)}{\partial r} \right]$$

$$dv = dr \qquad v = r$$

$$= i\hbar\, r^2 g(r) f^*(r) \Big|_0^{+\infty} - i\hbar \int_0^{+\infty} r g(r) f^*(r) dr - i\hbar \int_0^{+\infty} r^2 \frac{\partial g(r)}{\partial r} f^*(r) dr$$

$$\cancel{- i\hbar \int_0^{+\infty} r^2 g(r) \frac{\partial f^*(r)}{\partial r} dr} \; \cancel{+ i\hbar \int_0^{+\infty} g(r) \frac{\partial f^*(r)}{\partial r} r^2 dr}$$

$$= i\hbar\, r^2 g(r) f^*(r) \Big|_0^{+\infty} - i\hbar \Big[\int_0^{+\infty} f^*(r) \left(\frac{1}{r} + \frac{\partial}{\partial r} \right) g(r)\, r^2 dr$$

Da bi bilo ispunjeno

$$\langle f | \hat{p}_r | g \rangle = \left(\langle g | \hat{p}_r | f \rangle \right)^* \quad \text{mora da važi}$$

$$r^2 g(r) f^*(r) \Big|_0^{+\infty} = 0 \qquad \text{odnosno}$$

$$\lim_{r \to \infty} r\, g(r) = 0$$

40. Operator z-komponente momenta impulsa zadat je u koordinatnoj reprezentaciji, u sferno-polarnim koordinatama, kao $L_z = -i\hbar \frac{\partial}{\partial \varphi}$. Naći uslove njegove ermitičnosti na skupu funkcija definisanih na intervalu $[0, 2\pi]$.

Uslov da je $\hat{L}_z$ ermitsko

$$\langle f | \hat{L}_z | g \rangle = \left(\langle g | \hat{L}_z | f \rangle \right)^* \qquad (*)$$

$$\hat{I} = \int_0^{2\pi} |\varphi\rangle\langle\varphi| \, d\varphi \qquad \text{nepotrebno}$$

Direktno na koordinatnu reprezentaciju

$$\int_0^{2\pi} f^*(\varphi) \left(-i\hbar \frac{\partial}{\partial \varphi}\right) g(\varphi) d\varphi = \left[\int_0^{2\pi} g^*(\varphi) \left(-i\hbar \frac{\partial}{\partial \varphi}\right) f(\varphi) d\varphi \right]^*$$

$$= i\hbar \int_0^{2\pi} g(\varphi) \frac{\partial f^*(\varphi)}{\partial \varphi} d\varphi$$

$$u = g(\varphi) \quad \Rightarrow \quad du = \frac{\partial g(\varphi)}{\partial \varphi} d\varphi$$

$$dv = \frac{\partial f^*(\varphi)}{\partial \varphi} d\varphi \quad \Rightarrow \quad v = f^*(\varphi)$$

$$= i\hbar \, f^*(\varphi) g(\varphi) \Big|_0^{2\pi} - i\hbar \int_0^{2\pi} f^*(\varphi) \frac{\partial g(\varphi)}{\partial \varphi} d\varphi$$

da bi jednakost važila

$$f^*(2\pi) g(2\pi) = f^*(0) g(0) \Rightarrow f(0) = f(2\pi)$$

tj. samo ako su f-je periodične!